Automorphism Group of Graphs (Supplemental Material for Intro to Graph Theory)

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1 Introduction and Preliminary Results

In this supplement, we will assume that all graphs are undirected graphs with no loops or multiple edges. In graph theory, we talk about *graph isomorphisms*. As a reminder, an isomorphism between graphs G and H is a bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. A graph *automorphism* is simply an isomorphism from a graph to itself. In other words, an automorphism on a graph G is a bijection $\phi : V(G) \to V(G)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(G)$. This definition generalizes to digraphs, multigraphs, and graph with loops.

Let $Aut(G)$ denote the set of all automorphisms on a graph G. Note that this forms a *group* under function composition. In other words,

(i) $Aut(G)$ is closed under function composition.

(ii) Function composition is associative on $Aut(G)$. This follows from the fact that function composition is associative in general.

(iii) There is an identity element in $Aut(G)$. This is mapping $e(v) = v$ for all $v \in V(G)$.

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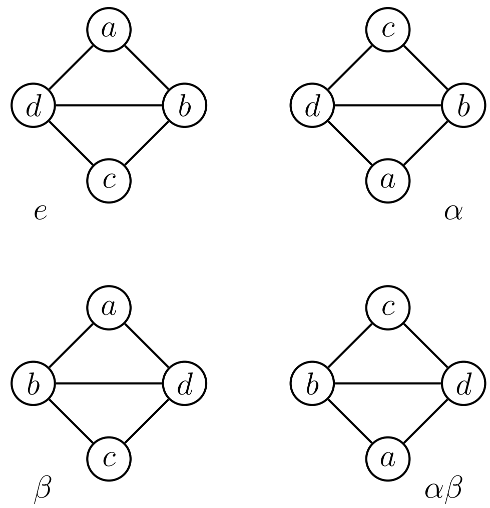

Figure 1: A graph and its automorphisms

(iv) For every $\sigma \in Aut(G)$, there is an <u>inverse</u> element $\sigma^{-1} \in Aut(G)$. Since σ is a bijection, it has an inverse. By definition, this is an automorphism.

Thus, $Aut(G)$ is the *automorphism group* of G. At this point, an example is order. Consider the graph G illustrated in Figure 1. An automorphism of G can leave every vertex fixed, this is the identity automorphism e. An automorphism of G can swap vertices a and c and leave the others alone. This is the automorphism $\alpha = (a, c)$. Similarly, we can swap vertices b and d while leaving a and c fixed. This results in the automorphism $\beta = (b, d)$. Finally, we can swap vertices a and c and swap vertices b and d. This results in the automorphism $\alpha\beta = (a, c)(b, d)$. It is easy to check that these are the only automorphisms. Hence, $Aut(G)$ is isomorphic to the Klein 4-group, $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

We can use the automorphism group to define a relationship between two vertices in G. Let $u, v \in V(G)$, vertex u *relates* to v if there exists $\phi \in Aut(G)$ such that $\phi(u) = v$. We claim that this is an *equivalence relation*.

(i) <u>Reflexive:</u> Note that $e(u) = u$ for all $u \in V(G)$, where e is the identity automorphism.

(ii) <u>Symmetric:</u> If $\phi(u) = v$, then $\phi^{-1}(v) = u$.

(iii) Transitive: If $\phi(u) = v$ and $\sigma(v) = w$, then $\sigma(\phi(u)) = w$.

Thus, the relationship is an equivalence relation. Like all equivalence relations, this induces a partition the vertex set into *equivalence classes*. These classes are usually called *automorphism classes*. If all of the vertices of the graph are in the same automorphism class, then we say that the graph is *vertex transitive*.

Some facts about the automorphisms of a graph.

Proposition 1.1 *[13, 23] Let G be a graph.*

(i) (Degree preserving) *For all $u \in V(G)$ and for all $\phi \in Aut(G)$, $deg(u) = deg(\phi(u))$.*

(ii) (Distance preserving) *For all $u, v \in V(G)$ and for all $\phi \in Aut(G)$, $d(u, v) = d(\phi(u), \phi(v))$.*

(iii) The automorphism group of G is equal to the automorphism group of the complement $\overline{G}$.

Proof. *(i)* Let $u \in V(G)$ with neighbors u_1,...,u_k. Let $\phi \in Aut(G)$. Since ϕ preserves adjacency, it follows that $\phi(u_1)$,...,$\phi(u_k)$ are neighbors of $\phi(u)$. Ergo, $deg(\phi(u)) \geq k$. If $v \notin \{u_1, ..., u_k\}$ is a neighbor of $\phi(u)$, then $\phi^{-1}(v)$ is a neighbor of u. Therefore, the neighbors of $\phi(u)$ are precisely $\phi(u_1)$,...,$\phi(u_k)$. Ergo, $deg(u) = deg(\phi(u))$.

(ii) Let $u, v \in V(G)$ and let $\phi \in Aut(G)$. Suppose that the distance from u to v is $d(u, v) = d$. Further, let $u = u_0, u_1, ..., u_{d-1}, u_d = v$ be a shortest path from u to v. Since ϕ preserves adjacency, $\phi(u) = \phi(u_0), \phi(u_1), ..., \phi(u_{d-1}), \phi(u_d) = \phi(v)$ is a path from $\phi(u)$ to $\phi(v)$. Thus, $d(\phi(u), \phi(v)) \leq d = d(u, v)$. Suppose that $\phi(u), v_1, ..., v_{m-1}, \phi(v)$ is a shortest path from $\phi(u)$ to $\phi(v)$. It follows that $u, \phi^{-1}(v_1), ..., \phi^{-1}(v_{m-1}), v$ is a shortest path form u to v. It follows that $d(u, v) \leq d(\phi(u), \phi(v))$. Hence, we have equality.

(iii) Note that automorphisms preserve not only adjacency, but non-adjacency as well. Hence, $\phi \in Aut(G)$ if and only if $\phi \in Aut(\overline{G})$. It follows that $Aut(G) = Aut(\overline{G})$. ∎

2 The Automorphism Group of Specific Graphs

In this section, we give the automorphism group for several families of graphs.

Let the vertices of the path, cycle, and complete graph on n vertices be labeled v_0, v_1,..., v_{n-1} in the obvious way.

Theorem 2.1 *(i) For all $n \geq 2$, $Aut(P_n) \cong \mathbb{Z}_2$, the second cyclic group.*

(ii) For all $n \geq 3$, $Aut(C_n) \cong D_n$, the nth dihedral group.

(iii) For all n, $Aut(K_n) \cong S_n$, the nth symmetric group.

Proof. *(i)* As in the proof of Proposition 1.1, any automorphism $\phi \in Aut(P_n)$ must either map a vertex of degree one to a vertex of degree one. Thus either $\phi(v_0) = v_0$ and $\phi(v_{n-1}) = v_0$ or $\phi(v_0) = v_{n-1}$. In either case, the orbit of the remaining vertices is precisely determined by their distance from v_0. In the first case, $\phi(v_i) = v_i$ for all i. This results in the identity automorphism. In the second case, $\phi(v_i) = v_{n-1-i}$ for all i. Thus, $Aut(P_n) \cong \mathbb{Z}_2$.

(ii) Consider the mapping $\rho(v_i) = v_{i+1}$, where the computation on the indices is computed modulo n. Since $v_i v_{i+1}$ is an edge in the graph, ρ is an automorphism. If n is even, then consider the mapping $\tau(v_i) = v_{n-1-i}$ and $\tau(v_{n-i-1}) = v_i$ for $i = 0, 1, ..., \frac{n}{2} - 1$. If n is odd, then consider the mapping $\tau(v_0) = v_0$, $\tau(v_i) = v_{n-i}$, and $\tau(v_{n-i}) = v_i$ for $i = 1, ..., \frac{n-1}{2}$. In both cases, $v_i v_{i+1}$ and $v_{n-1-i} v_{n-2-i}$ are both edges in C_n. Thus, τ is an automorphism. Note that $\rho^n = \tau^2 = e$ and $\rho^k \tau = \tau \rho^{n-k}$. Hence ρ and τ generate the nth dihedral group, D_n. Since we can think of C_n as a regular n-gon, we have that $Aut(C_n) \cong D_n$.

(iii) Since S_1 is the trivial group, the result holds for $n = 1$. For the remainder of the proof, let $n \geq 2$. Let x and y be distinct vertices of K_n. Consider the mapping $\phi(x) = y$, $\phi(y) = x$, and $\phi(v) = v$ for all other $v \in V(K_n)$. Since x and y are both adjacent to every vertex, ϕ is an automorphism of K_n. Thus, every transposition of two vertices is an automorphism. Since the set of all transpositions generates S_n, the result follows. ■

The *complete bipartite graph* $K_{n,m}$ has $V(K_{n,m}) = X \cup Y$, where $X = \{x_1, ..., x_n\}$, $Y = \{y_1, ..., y_m\}$, and $X \cap Y = \emptyset$. The edge set of this graph is $E(K_{n,m}) = \{x_i y_j : i = 1, ..., n, j = 1, ..., m\}$.

Theorem 2.2 *For the complete bipartite graph, $K_{n,m}$, where $n \geq m$:*

(i) If $n > m$*, then* $Aut(K_{n,m}) \cong S_n \times S_m$*.*

(ii) If $n = m$*, then* $Aut(K_{n,m}) \cong S_n^2 \ltimes \mathbb{Z}_2$*.*

Proof. By Theorem 2.1, $Aut(K_n) \cong S_n$. By Proposition 1.1, it follows that $Aut(\overline{K_n}) \cong S_n$. Thus, any automorphism of the form (x_i, x_j) or of the form (y_k, y_ℓ) is in $Aut(K_{n,m})$. Thus, $S_n \times S_m$ is a subgroup of $Aut(K_{n,m})$.

(i) Suppose that $n > m$. Since $deg(x_i) = m$ and $deg(y_j) = n$, it follows from the proof of Proposition 1.1 that there is no automorphism ϕ such that $\phi(x_i) = y_j$. Thus, $Aut(K_{n,m}) \cong S_n \times S_m$.

(ii) Suppose that $n = m$. Here, it is possible to map elements of X to elements of Y. Since every element of X is adjacent to every element Y, if we map one element of X, then we must map *every* element of X must be mapped to a distinct element of Y. Such a mapping will be its own iverse. Thus, in addition to the automorphisms described in (i), we also have automorphisms of the form $\prod_{i=1}^{n}(x_i, y_{\pi(i)})$, where π is a permutation on the set $\{1, ..., n\}$. Thus, the automorphism group is generated by (x_i, x_j), (y_k, y_ℓ), and $\prod_{i=1}^{n}(x_i, y_i)$. Thus, the automorphism group is isomorphic to $S_n^2 \ltimes \mathbb{Z}_{\not{2}}$.
■

The *double star* is the tree with two adjacent non-leaf vertices x and y such that $x_1, ..., x_n$ are the leafs adjacent to x and $y_1, ..., y_m$ are the leafs adjacent to y. This graph is denoted $S_{n,m}$.

Theorem 2.3 *For the double star* $S_{n,m}$*, where* $n \geq m$*:*

(i) If $n > m$*, then* $Aut(S_{n,m}) \cong S_n \times S_m$*.*

(ii) If $n = m$*, then* $Aut(S_{n,m}) \cong S_n^2 \ltimes \mathbb{Z}_2$*.*

Proof. Note that any element of the set $\{x_1, ..., x_n\}$ can be mapped to any other element of the same set. Likewise, any element of the set $\{y_1, ..., y_m\}$ can be mapped to an element of $\{y_1, ..., y_m\}$. These permutations result in a subgroup of $Aut(S_{n,m})$ that is isomorphic to $S_n \times S_m$. If $n \neq m$, then x and y have different degrees. Thus, these are the only automorphisms possible. Thus, (i) holds.

If $n = m$, then we can map x to y. However, as their leaves will be carried along in this mapping, each x_i must be mapped to some y_j. Thus, the group of permutations is generated by (x_i, x_j), (y_k, y_ℓ), and $(x, y)\prod_{i=1}^{n}(x_i, y_i)$. Thus, the automorphism group is isomorphic to $S_n^2 \ltimes \mathbb{Z}_2$. ■

The *Petersen Graph* is one of the most important graphs. In fact, entire books have been written about the Petersen graph [16]. The Petersen graph $K(5, 2)$ is the graph where the vertex set is all 2-element subsets of $\{1, 2, 3, 4, 5\}$. Two vertices are adjacent in $K(5, 2)$ if and only if their 2-sets are disjoint.

Theorem 2.4 *For the Petersen graph $K(5, 2)$, $Aut(K(5, 2)) \cong S_5$.*

Proof. Let $\pi \in S_5$. This induces a permutation on the 2-element subsets of [5] that make up the vertex set. Namely, $\pi^{(2)}(\{x, y\}) = \{\pi(x), \pi(y)\}$. Clearly, $\pi^{(2)}$ is a bijection on the vertex set. Further, $\{x, y\}$ and $\{w, z\}$ are disjoint if and only if $\{\pi(x), \pi(y)\}$ and $\{\pi(w), \pi(z)\}$ are disjoint. Ergo, $\pi^{(2)}$ is an automorphism. If σ is any other automorphism of the Petersen graph, then σ must permute the 2-element subsets in such a way to preserve adjacency. Thus, $\sigma = \pi^{(2)}$ for some $\pi \in S_5$. Therefore, $Aut(G(5, 2)) \cong S_5$. ■

The Petersen graph is a special case of *Kneser graphs*. The Kneser graph $K(n, k)$ has as its vertex set all k-element subsets of $\{1, ..., n\}$. Two vertices in $K(n, k)$ are adjacent if and only if their k-sets are disjoint. Using a similar argument as in Theorem 2.4, we can show that $Aut(G(n, k))$ is isomorphic to the nth symmetric group S_n. For more details on this proof, refer to [3, 15].

Note that the Kneser graph is *not* what people call a "generalized Petersen graph" in the literature. Usually, when mathematicians refer to a generalized Petersen graph, they are referring to the family of graphs introduced by Coxeter [8]. However, their name is due to Watkins [22]. The automorphism group of the generalized Petersen graphs was determined in [12].

3 Cartesian Products

Recall that the *Cartesian product* of graphs G and H is the graph with vertex set $\{(g, h) : g \in V(G), h \in V(H)\}$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either *(i)* $g_1 = g_2$ and $h_1h_2 \in E(H)$ *or (ii)* $h_1 = h_2$ and $g_1g_2 \in V(G)$. This graph is denoted $G\Box H$[1]. For additional references

[1] The $G\Box H$ notation is consistent with West [24] and most of the literature. However, Buckley and Lewinter [4] use the notation $G \times H$ for this same product.

on Cartesian products, refer to [14, 17, 18, 21]. Note that this product can be generalized to an arbitrary number of graphs.

Theorem 3.1 *Let G and H be graphs. It follows that $Aut(G) \times Aut(H)$ is a subgroup of $Aut(G \Box H)$.*

Proof. Let $\phi \in Aut(G)$ and let $\theta \in Aut(H)$. Consider the mapping $\xi : V(G \Box H) \to V(G \Box H)$ defined by $\xi((g,h)) = (\phi(g), \theta(h))$. We claim that ξ is an automorphism of $G \Box H$. Suppose that (g_1, h_1) and (g_2, h_2) are adjacent in $G \Box H$. If $h_1 = h_2$, then $\theta(h_1) = \theta(h_2)$. Further, g_1 and g_2 would be adjacent in G. It follows that $\phi(g_1)$ and $\phi(g_2)$ are adjacent in G. Since $\xi((g_1, h_1)) = (\phi(g_1), \theta(h_1))$ and $\xi((g_2, h_2)) = (\phi(g_2), \theta(h_2))$, it follows that $\xi((g_1, h_1))$ and $\xi((g_2, h_2))$ are adjacent in $G \Box H$. A similar argument holds if $g_1 = g_2$ and h_1 is adjacent to h_2 in H. Thus, $\xi \in Aut(G \Box H)$ and the result follows. ■

A natural question is when $Aut(G \Box H)$ contains an element that is not of the form described in Theorem 3.1. To do this, we need a bit more terminology.A graph D is a *divisor* of a graph G if their exists a graph H such that $G \cong D \Box H$. A graph P is *prime* if P has no divisor other than itself and K_1[2]. Graphs G and H are *relatively prime* if they share no common factor other than K_1. With these terms in mind, we present results about the automorphism group of Cartesian products.

Theorem 3.2 *[21, 23]* (i) *Every connected graph G can be written as $G \cong G_1 \Box \cdots \Box G_k$, where the G_i are prime graphs. This factorization is unique, up to permutations on the prime factors*[3]. (ii) *If G is a connected graph, then $Aut(G)$ is generated by $Aut(G_i)$ and the transpositions interchanging isomorphic prime divisors.* (iii) *In particular, if the G_i are relatively prime connected graphs, then $Aut(G)$ is the direct product of the $Aut(G_i)$ over all i.*

The comment in Theorem 3.2 about "transpositions interchanging isomorphic prime divisors" deserves a bit more explanation. Suppose that the connected graph G has prime factorization $G = G_1 \Box \cdots \Box G_n$, where G_i and

[2]Examples of prime graphs include trees, odd cycles, and complete graphs.

[3]The factorization may not be unique for disconnected graphs. As an example, note that $(K_1 \cup K_2 \cup K_2^2) \Box (K_1 \cup K_2^3)$ is isomorphic to $(K_1 \cup K_2^2 \cup K_2^4) \Box (K_1 \cup K_2)$.

G_j are isomorphic prime divisors for some $i \neq j$. Thus, there is isomorphism $\psi : G_i \to G_j$. For all $k \in \{1, ..., |V(G_i)|\}$, suppose that $v_{i,k} \in V(G_i)$ and $v_{j,k} \in V(G_2)$ such that $\psi(v_{i,k}) = v_{j,k}$. Then there is an automorphism $\zeta \in Aut(G_i \Box G_j)$ such that $\zeta = (v_{j,1}, v_{i,1})...(v_{j,|V(G_2)|}, v_{i,|V(G_1)|})$. In other words, ζ "swaps" the isomorphic prime factors G_i and G_j.

By Theorem 3.2 (iii), if G and H are relatively prime, then $Aut(G \Box H) \cong Aut(G) \times Aut(H)$. Hence, all automorphisms are of the form described in the proof of Theorem 3.1. An immediate consequence of Theorem 3.2 is given in the following corollary.

Corollary 3.3 *For the hypercube Q_n, $Aut(Q_n) \cong \mathbb{Z}_2^n \ltimes S_n$.*

4 Frucht's Theorem

In the previous sections, we discussed the automorphism group of various graphs. In this section, we consider an alternative problem proposed by König in 1936 [19]. Suppose that we are given a finite group Γ[4]. Our goal is to find a graph G such that $Aut(G) \cong \Gamma$. The result was proven by Frucht in 1939 [10]. The proof of Frucht's Theorem involves use of the Cayley graph (introduced in 1878 [5]). Recall that the Cayley digraph $Cay(\Gamma, S)$ has a vertex for each element of the group Γ. Let $x, y \in V(Cay(\Gamma, S)$. There is a arc pointing from x to y if and only if there exists an $g \in S$ such that $xg = y$. Traditionally, the different generators are represented by different colored arcs. For more information on the Cayley digraph, refer to the relevant section in Fraleigh [9]. Our treatment of Frucht's Theorem will follow that of Chartrand, Lesniak, and Zhang [6].

We begin with some terminology. Let $\phi \in Aut(Cay(\Gamma, S))$. We say that ϕ is *color-preserving* if for every arc (x, y) in $Cay(\Gamma, S)$, the arcs (x, y) and $(\phi(x), \phi(y))$ have the same color. The following proposition is straightforward to prove using the techniques in the senior-level algebra course.

Proposition 4.1 *The set of color-preserving automorphisms is a subgroup of $Aut(Cay(\Gamma, S))$.*

A useful characterization of color-preserving automorphisms is given in the next theorem.

[4]Fraleigh [9] uses G to denote a group. However, we have been using G to denote a graph. Hence to keep levels of abstraction sufficiently clear, we use Γ to denote the group.

Theorem 4.2 *Let Γ be a finite group with generating set S. Let ϕ be a permutation of $V(Cay(\Gamma, S))$. The permutation ϕ is a color-preserving automorphism of $Cay(\Gamma, S)$ if and only if $\phi(gh) = (\phi(g))h$ for every $g \in \Gamma$ and $h \in S$.*

The significance of color-preserving automorphisms is given in the next theorem.

Theorem 4.3 *Let Γ be a finite group with generating set S. The group of color-preserving automorphisms of $Cay(\Gamma, S)$ is isomorphic to Γ.*

Proof. Let $\Gamma = \{g_1, ..., g_n\}$. For $i = 1, ..., n$, define $\phi_i : V(Cay(\Gamma, S)) \to V(Cay(\Gamma, S))$ by $\phi_i(g_s) = g_i g_s$. Since Γ is a group, ϕ_i is one-to-one and onto. Let $h \in S$. Then for each i, $1 \leq i \leq n$, and for each s, $1 \leq s \leq n$,

$$\phi_i(g_s h) = g_i(g_s h) = (g_i g_s)h = \phi_i(g_s)h.$$

Thus, ϕ_i is a color-preserving automorphism of $Cay(\Gamma, S)$ by Theorem 4.2.

Let ϕ be an arbitrary color-preserving automorphism of $Cay(\Gamma, S)$. Let $e = g_1$ be the identity element of Γ. Suppose that $\phi(g_1) = g_r$. Let $g_s \in \Gamma$. By definition, we can write g_s as a product of generators. In other words, $g_s = h_1...h_t$, where $h_j \in S$ for $j = 1, ..., t$. Ergo,

$$\phi_1(g_s) = \phi(g_1 h_1...h_t) = \phi(g_1 h_1...h_{t-1})h_t$$

$$= \phi(g_1 h_1...h_{t-2})h_{t-1}h_t = \cdots = \phi(g_1)h_1...h_t = g_r g_s.$$

Thus $\phi = \phi_r$.

We now show that the mapping θ defined by $\theta(g_i) = \phi_i$ is an isomorphism from Γ to the color-preserving automorphisms of $Cay(\Gamma, S)$. Since θ is one-to-one and onto, we need only show that it is a homomorphism. Let $g_i g_j = g_k$. Then $\theta(g_i g_j) = \theta(g_k) = \phi_k$ and $\theta(g_i)\theta(g_j) = \phi_i \phi_k$. It follows that

$$\phi_k(g_s) = g_k g_s = (g_i g_j)g_s = g_i(g_j g_s) = \phi_i(g_j g_s) = \phi_i(\phi_j(g_s)) = (\phi_i \phi_j)g_s.$$

∎

The idea of the construction is rather simple. Namely, we replace each arc in the Cayley graph with a undirected graph that still indicates the direction of the original arc. This can be done by replacing the arc with a path on four vertices and appending a path to one of the center vertices of the path. Such

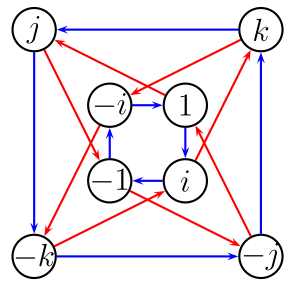

Figure 2: Cayley graph $Cay(Q_8, \{i, j\})$

a graph only admits the identity automorphism, so no new symmetries are introduced. In the case of involutions (represented by undirected edges in the Cayley graph) can be replaced with a path, with a single subpath appended onto it. Different colors of arcs can be differentiated by appending different lengths of paths. In other words, we preserve the original symmetries of the Cayley graph without introducing any new symmetries.

Theorem 4.4 (Frucht's Theorem [10]) *Given a finite group* Γ*, there exists a graph* G *such that* $Aut(G)$ *is isomorphic to* Γ.

Proof. Let Γ be a finite group and let $S = \{g_1, ..., g_k\}$ be a generating set of Γ. Suppose that for some $x, y \in \Gamma$ and $g_i \in S$, we have that $xg_i = y$. Thus, (x, y) is an arc "colored" g_i in $Cay(\Gamma, S)$. By Theorem 4.3, the color-preserving automorphisms of $Cay(\Gamma, S)$ is isomorphic to Γ. To transform $Cay(\Gamma, S)$ to the required graph, we delete the arc (x, y) and replace it with the path x, $u_{x,y}$, $u'_{x,y}$, y. At $u'_{x,y}$, we construct a path P'_{g_i} of length i. Each path corresponding to "color" g_i is distinguished by the lengths of the subpath P'_{g_i}. The differing lengths of the paths appended to $u_{x,y}$ and $u'_{x,y}$ preserves the direction of the arc (x, y) in the Cayley graph. This construction is repeated for every arc in $Cay(\Gamma, S)$. Denote the resulting graph G.

We claim that $Aut(G)$ is isomorphic to Γ. Let $\phi \in Aut(G)$ and $u \in V(G)$. If u is an endpoint of a subpath of the form P'_{g_i}, then $\phi(u)$ is also an endpoint of a subpath of the form P'_{g_i}. A similar argument holds for all vertices along these subpaths. Since these edges of the color g_i in G corresponds to the generator g_i in Γ, it follows that every automorphism $\phi \in Aut(G)$ is an element of Γ. Thus, $Aut(G) \cong \Gamma$. ■

Example 4.5 *As an example of this construction, consider the quaternion group* $Q_8 = \{\pm 1, \pm i, \pm j, \pm k : i^2 = j^2 = k^2 = ijk = -1\}$. *A generating*

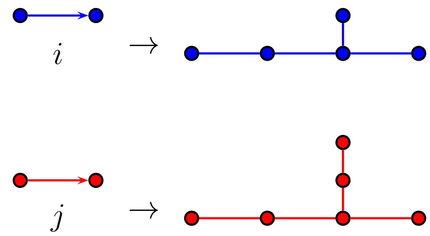

Figure 3: Replacements for Frucht's Theorem

set for this group is $S = \{i, j\}$. *In Figure 3, we give the Cayley graph,* $Cay(Q_8, \{i, j\})$. *For emphasis, we have colored the arcs in the graph corresponding to* i *and* j *blue and red, respectively. We replace the arcs as shown in Figure* **??**. *The result is the graph shown in Figure 4.*

Example 4.6 *Consider the Alternating group* A_4. *This is the group of order 12 consisting of all even permutations on the set* $\{1, 2, 3, 4\}$. *This group is generated by* $(1, 2, 3)$ *and* $(1, 2)(3, 4)$. *We represent right multiplication by* $(1, 2, 3)$ *as a blue arc. We represent right multiplication by* $(1, 2)(3, 4)$ *as a red edge. Note that since* $(1, 2)(3, 4)$ *is its own inverse, the red edges are undirected. The resulting Cayley graph is illustrated in Figure 5*[5] *When replacing our arcs, we replace the blue arcs as above. In the case of the red edges, we can simply replace them with single edges as there is no orientation to preserve*[6]*. The resulting graph is given in Figure 6.*

It turns out that graphs are rather pliable things. For this reason, Frucht's Theorem still holds, even if we restrict our attention to graphs that have a specified properties. Examples of such results include:

(i) Given a finite group Γ, there is a k-regular graph G such that $Aut(G) \cong \Gamma$ [11, 20].

(ii) Given a finite group Γ, there is a k-vertex-connected graph G such that $Aut(G) \cong \Gamma$ [20].

[5] Note that this is the same Cayley graph that appears on the cover of Fraleigh [9].

[6] In the case where more than one of our generators is an involution, we can replace each involution with a symmetric graph. However, each graph must be distinct so that we can distinguish the generators.

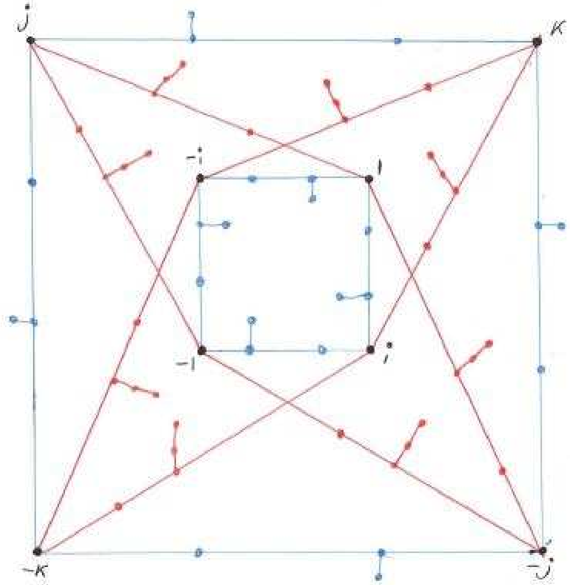

Figure 4: A graph whose automorphism group is isomorphic to Q_8

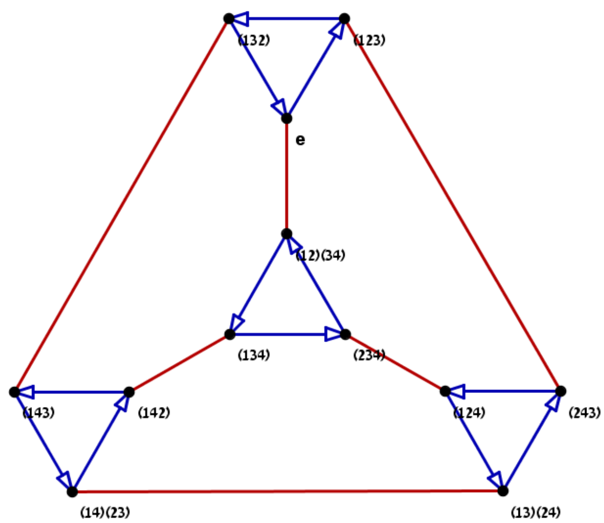

Figure 5: The Cayley graph $Cay(A_4, \{(1,2,3),(1,2)(3,4)\})$

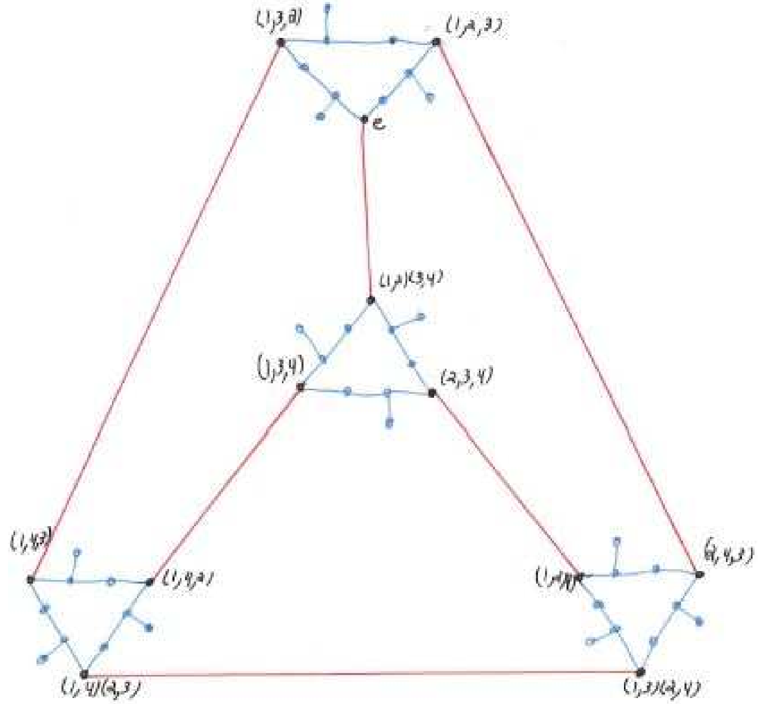

Figure 6: A graph whose automorphism group is isomorphic to A_4.

(iii) Given a finite group Γ, there is a k-chromatic graph G such that $Aut(G) \cong \Gamma$ [20].

Note that any of the Frucht-type constructions will produce a graph that has many more vertices than elements in the target group. Thus, a natural question is the following: Given a finite group Γ, find a graph G such that:

(i) The automorphism group of G is isomorphic to Γ.

(ii) Among all graph whose automorphism group is isomorphic to Γ, G has the minimum number of vertices.

5 Related Ideas

There are several ideas related to the automorphism group. Any of these ideas could be the basis for a entire supplement. This being the case, we only introduce these ideas and provide the relevant reference.

In 1996, Albertson and Collins [1] introduced the *distinguishing number of a graph*. For the distinguishing number, we label the vertices of G with (not-necessarily distinct) elements of $\{1, ..., k\}$. The goal is to do this in such a way that no element of $Aut(G)$ preserves all of the vertex labels. However, we wish to do this in such a way that we use the minimum number of labels as possible. This minimum number is the *distinguishing number*. For example, we can label the first vertex of the path 1 and the remaining vertices 2. The first vertex is clearly distinguished as it is the only one labeled 1. The remaining vertices are also distinguished by their distance from the unique vertex labeled 1. Thus, the distinguishing number of the path is 2.

In 2006, this was followed by a paper by Collins and Trenk [7] that introduced the *distinguishing chromatic number of a graph*. The idea is that we assign numbers to the vertices in order to break the symmetries of the graph. However, if two vertices are adjacent, then they must receive different labels. Again, the goal is to use the minimum number of labels possible. This minimum number is the *distinguishing chromatic number*. For example, the distinguishing chromatic number of the path P_n is 2 when n is even and 3 when n is odd.

In 2017, I submitted a paper [2] that introduced the notion of a *palindromic labeling*. A palindromic labeling is a bijection $f : V(G) \to \{1, ..., |V(G)|\}$ such that if $uv \in E(G)$, then there exists $x, y \in V(G)$ such that $xy \in E(G)$,

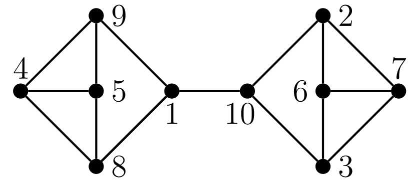

Figure 7: A palindromic labeling on a graph

$f(x) = |V(G)| + 1 - f(u)$, and $f(y) = |V(G)| + 1 - f(v)$. An example of a palindromic labeling on a graph is given in Figure 7. A graph that admits a palindromic labeling is a *palindromic graph*. Examples of palindromic graphs include paths, cycles, and complete graphs. Equivalently, a graph G is palindromic if there exists $\phi \in Aut(G)$ such that ϕ^2 is the identity and ϕ has at most one fixed point.

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