

Cops and Robbers on Graphs

(A Supplement for Intro to Graph Theory)

Robert A. Beeler*

January 8, 2016

1 Introduction

The game of *cops and robbers* on graphs is a combinatorial game introduced independently by Quilliot [20, 21, 22] and by Nowakowski and Winkler [17]. This is an example of a graph *searching* game (see [2, 7, 10, 18, 19] for a more on such games). Our treatment of the game of cops and robbers will be similar to the survey articles by Aigner and Fromme [1] and Hahn [11] as well as the book by Bonato and Nowakowski [5]. For more information on combinatorial games, refer to [3, 8, 23, 24]. For more information on general games of strategy, refer to [26, 27, 28].

There are a number of variations on the game of cops and robbers on graphs. However, we will be concerned with the original version. In this version, a graph G is given. There are two players, a *cop* and a *robber*. The cop chooses a vertex C . Then, the robber chooses a vertex, R . Players move alternate moves, beginning with the cop. A *move* consists of staying at one's present vertex or moving to an adjacent vertex. Both players have *perfect information*. In other words, each player knows the position and moves of the other player at all times. The cop wins if manages to occupy the same vertex as the robber. This is referred to as a *capture*. The robber wins if he is able to avoid this forever. Note that based on the above definition, we *only* consider connected graphs.

*Department of Mathematics and Statistics, East Tennessee State University, Johnson City, TN 37614-1700 USA email: beelerr@etsu.edu

Using the Fundamental Theorem of Combinatorial Games [27] (this is also known as von Neumann's Theorem [26]), either the cop or the robber has a *winning strategy* on any given graph. A winning strategy is one of the options available to the player that will *guarantee* victory for that player, regardless of the actions of their opponent. Graphs in which the cop has a winning strategy are called *cop-win* graphs. Conversely, graph in which the robber has a winning strategy are called *robber-win* graphs.

2 Cop-win graphs

The goal of this section is to classify those graphs which are cop-win graphs. We begin with some motivating examples.

Proposition 2.1 *If G is a graph with a universal vertex, then G is a cop-win graph.*

Proof. Let G be a graph with universal vertex u . By definition, u is adjacent to every vertex in G . Thus, on their first turn, the cop moves to the vertex occupied by the robber. Hence, the cop captures the robber on their first move. ■

Theorem 2.2 *If G is a tree, then G is a cop-win graph.*

Proof. Let G be any tree. The cop chooses any vertex initially. Since the deletion of any non-leaf vertex disconnects a tree, this separates the graph into two components. The robber's initial location must be in one of these components. Let the component occupied by the robber be denoted \mathcal{R} . On each turn, the cop moves towards the robber. This again separates the graph into two components. More importantly, the number of vertices in \mathcal{R} has decreased by one. Hence, on each subsequent move by the cop, the number of options available to the robber decreases by at least one. Ergo, the cop will eventually capture the robber. ■

Theorem 2.3 *If G is a cycle of length at least 4, then G is a robber-win graph.*

Proof. Let G be a cycle of length n , where $n \geq 4$. For purposes of exposition, we label the vertices of the cycle with the integers modulo n . Since the cycle is vertex transitive, we may assume that the cop chooses vertex 0 for their starting position. We need only give the robber's winning strategy. Suppose that the robber chooses vertex 2 initially. If the cop moves to vertex i , then the robber moves to vertex $i + 2 \pmod{n}$. Notice that this strategy allows the robber to maintain a distance of 2 away from the cop. Hence, the robber can evade pursuit forever. ■

Corollary 2.4 *Suppose that G is a graph with a vertex induced subgraph H , where H is a cycle of length at least 4. It follows that G is a robber-win graph.*

Proof. Suppose that the cop takes vertex C on their first turn. For their initial position, the robber takes any vertex v on H such that the distance between C and v is at least 2. The robber then moves (or remains stationary) in such a way that the distance between them and the cop remains at least 2. Hence, the robber has a winning strategy for the same reasons as Theorem 2.3. ■

In Theorem 2.2, the ability of the cop to separate the graph into smaller components was crucial to their winning strategy. This notion is formalized with the idea of a *retract*. Let H be a vertex induced subgraph of a graph G . Let f be a function mapping $V(G)$ onto $V(H)$ such that if $xy \in E(G)$, then $f(x)f(y) \in E(H)$ (in other words, f is a graph *homomorphism*). The function f is a *retraction* of G onto H if $f(x) = x$ for all $x \in V(H)$. If such a function exists, then we say that H is a *retract* of G .

At this point, an example is order. Consider the graph given in Figure 1. We define the function $f : V(G) \rightarrow V(H)$ as follows $f(a) = f(b) = b$, $f(c) = c$, $f(d) = d$, and $f(e)$. Clearly, If $x \in V(H)$, then $f(x) = x$. We need only check that f is a graph homomorphism. Since only a is affected by f , we need only check the edges ab and ae . Note that $f(a)f(e) = be \in E(H)$. Further, $f(a)f(b) = bb \in E(H)$ (it is convenient in such cases to assume that every vertex has a loop). Thus, the graph H is a retraction of G .

Theorem 2.5 [4] *Let G be a cop-win graph. If H is a retract of G , then H is a cop-win graph.*

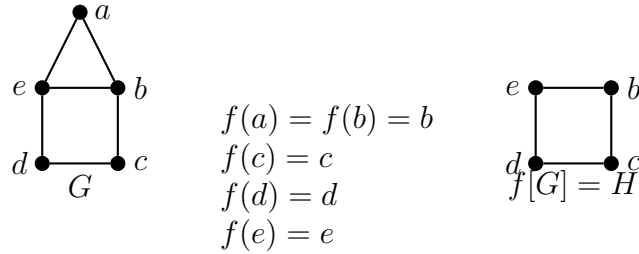


Figure 1: A graph and its retraction

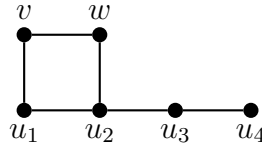


Figure 2: A counter-example to the converse of Theorem 2.5

Proof. Suppose that the cop has a winning strategy on G . Let $f : V(G) \rightarrow V(H)$ be a retraction on G . Consider two parallel games. The first game is played on G , while the second is played on H . In G , the cop plays as usual. In H , the cop plays as an image of the cop in G . In other words, if the cop moves from x to y in G , then the cop moves from $f(x)$ to $f(y)$ in H . Since f preserves edges, this is well-defined.

We claim that this gives a winning strategy on H . Suppose that the robber is at vertex R , while the cop is at vertex C . The cop plays on G with R restricted to H . Suppose that the cop is about to win in G . It must be that R and each neighbor $v \in V(H)$ (as well as the neighbors of R in $G - H$) are adjacent to C . But then the edge RC becomes the edge $Rf(C)$ under the retraction and vC becomes $vf(C)$. Therefore, the robber loses in the game played in H in the next round. ■

A natural question is to whether the converse of Theorem 2.5 is true. In other words, suppose that H is cop-win graph and H is a retraction of G . Will G always be a cop-win graph? Consider the graph given in Figure 2. We obtain the graph P_4 with the retraction $f(v) = f(u_1) = u_1$, $f(w) = f(u_2) = u_2$, $f(u_3) = u_3$, and $f(u_4) = u_4$. Note that H is a tree, so it is a cop-win graph. However, G is robber-win because it contains an induced C_4 .

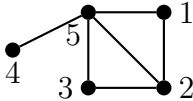


Figure 3: A cop-win ordering on a graph

Another useful idea is that of a *corner*. Consider the graph G given in Figure 1. Suppose that the robber is at vertex a on this graph while the cop is at vertex b . Recall that the *closed neighborhood* of a vertex x in G is the set $N[x] = \{w : xw \in E(G)\} \cup \{x\}$. In Figure ??, every vertex in $N[a]$ is in $N[b]$. Thus, the cop can win on their next move, regardless of the the actions of the robber. So, the robber is cornered at a . To formalize, a vertex u is a *corner* of a graph G if there is a vertex v such that $N[u] \subseteq N[v]$. In this case, the vertex v *covers* the vertex u . Note that every leaf of a tree is a corner. Likewise, a universal vertex covers every other vertex in a graph.

Lemma 2.6 [17] *If G is a cop-win graph, then G contains at least one corner.*

Proof. Let G be a cop-win graph. Consider the second to last move of the cop. Further, suppose that the robber is at vertex R while the cop is at vertex C . The robber could remain at vertex R , so vertex C must be adjacent to vertex R . Likewise, the robber could move to any neighbor of R . Hence, C must be adjacent to every neighbor of R . In other words, $N[R] \subseteq N[C]$. Ergo, R is a corner by definition. ■

Corners play a vital role in our goal of classifying the cop-win graphs. A graph is *dismantlable* if some sequence of deleting corners results in a single vertex. Note that every tree is dismantlable, just repeatedly delete leaves. Observe that such graphs have a recursive structure which can be defined as a *cop-win ordering* on a graph G of order n . Such an ordering is a labeling of the vertices of G using the elements of $\{1, \dots, n\}$ in such a way that for each $i < n$, the vertex i is a corner in the subgraph induced by $\{i, i + 1, \dots, n\}$. This labeling indicates the order in which vertices are deleted, namely in ascending order of their labels. Figure 3 gives an example of a cop-win graph and its associated cop-win ordering.

We are now prepared to proceed with our characterization of cop-win graphs.

Theorem 2.7 *A graph is cop-win if and only if it is dismantlable.*

Proof. Let G be a cop-win graph on n vertices. We proceed by induction on n . Note that when $n = 1$, G is only a single vertex. So, the claim holds trivially. Assume that for some $n \geq 1$, all cop-win graphs on n vertices are dismantlable. Let G' be a cop-win graph on $n + 1$ vertices. By Lemma 2.6, G' has a corner u which is covered by vertex v . Note that $G' - \{u\}$ is a retract of G' . Hence, $G' - \{u\}$ is a cop-win graph by Theorem 2.5. Hence, the result follows by the Principle of Mathematical Induction.

Suppose that G is a dismantlable graph of order n . Again, we proceed by induction on n . If $n = 1$, then G is trivially a cop-win graph. Assume that for some $n \geq 1$, that if G is a dismantlable graph on n vertices, then G is cop-win. Let G' be a dismantlable graph on $n + 1$ vertices. By definition, G' has a corner u which is covered by v . Note that $G' - \{u\}$ is a dismantlable graph on n vertices. Hence, it is cop-win by inductive hypothesis. It suffices to give the cop's winning strategy on G' . The cop plays on $G' - \{u\}$ using his winning strategy there. However, whenever the robber moves to u , then cop moves as though the robber moves to v . This is well-defined because $uv \in E(G')$. Think of the vertices of $G' - \{u\}$ as images under the retraction f which maps u to v and fixes all other vertices. Now the cop eventually captures the image of the robber $f[R]$ with his winning strategy on $G' - \{u\}$. If $R = f[R]$, then the robber is captured. Otherwise, the robber is on u with the cop on v . Since v covers u , the cop wins on the next turn. ■

3 The cop number of a graph

As shown in the previous section, the cop-win graphs have been classified. Thus a natural question is to determine the minimum number of cops that will *guarantee* that the robber is captured on graph G , assuming perfect strategy. This minimum number, called the *cop number of G* , was introduced by Berarducci and Intrigila [4]. The cop number of G is denoted $c(G)$. Our goal in this section is to give several results regarding the cop number. When considering the cop number, it is important to note the following:

- (i) Any k -subset of the cops can move on their turn, while the rest remain stationary.

- (ii) All k cops move simultaneously on their turn.
- (iii) Any number of cops may occupy a single vertex.

Note that for all graphs, we could simply place a cop on every vertex. Hence $c(G) \leq n$ for all graphs G on n vertices. We can obtain a less trivial upper bound using the *domination number* of G . Recall that $S \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G)$, either $v \in S$ or v is adjacent to a vertex in S (see [12] for more on domination in graphs). The cardinality of the minimum dominating set on G is denoted $\gamma(G)$.

Proposition 3.1 *Let G be a graph. The cop-number of G satisfies $c(G) \leq \gamma(G)$.*

Proof. Let S be a minimum dominating set of G . Initially, the cop takes all vertices in S . By definition of S , every vertex of G is either in S or adjacent to a vertex in S . Hence, regardless of where the robber starts, they will be captured on the first turn. Ergo, at most $|S| = \gamma(G)$ cops are sufficient to catch the robber. ■

The following result is obvious based on our definitions.

Proposition 3.2 *The graph G is cop-win if and only if $c(G) = 1$.*

Likewise, we can determine the cop number for cycles.

Corollary 3.3 *Let C_n be a cycle on n vertices, where $n \geq 4$. The cop number for C_n is $c(C_n) = 2$.*

Proof. By Theorem 2.3, one cop is insufficient to catch the robber. Thus we need only give the strategy for two cops to capture the robber. The cop chooses vertices 0 and $\lfloor n/2 \rfloor$. This separates the cycle into two components. On their turn, the robber chooses a vertex in one of these two components. We denote this component \mathcal{R} . On each turn, the both cops move towards the robber. This reduces the number of vertices in \mathcal{R} by two. Hence, one each turn, the robber has two less options. Ergo, the cops will eventually capture the robber. ■

The proof of the following theorem is the same as in Theorem 2.5.

Theorem 3.4 [4] *If H is a retraction of G , then $c(H) \leq c(G)$.*

A similar, yet more complicated bound is given below.

Theorem 3.5 [4] *If H is a retract of G , then*

$$c(G) \leq \max\{c(H), c(G - H) + 1\}.$$

Proof. Let $m = \max\{c(H), c(G - H) + 1\}$. Let $f : V(G) \rightarrow V(H)$ be a retraction. Since $m \geq c(H)$, the cops play by using the cops' winning strategy in H and capture the robber's image $f[R]$ in H . These moves also occur in G , since f is a graph homomorphism. So, if the robber is in H , then $R = f[R]$ and the cops win. It follows that $c(G) \leq c(H)$.

Otherwise, the robber is in $G - H$. One cop protects H by occupying the image $f[R]$. Since f is a homomorphism, this is always possible. Thus, the robber must remain in $G - H$ to avoid capture. At most $c(G - H)$ additional cops are needed to capture the robber in $G - H$. Thus, $c(G) \leq c(G - H) + 1$.

■

Recall that a graph is *planar* if it can be drawn in the plane in such a way that no edges cross. By Kuratowski's Theorem [13], a graph is planar if and only if it contains no subdivision of $K_{3,3}$ or K_5 as a subgraph.

Theorem 3.6 [1] *If G is a planar graph, then $c(G) \leq 3$.*

Previously, we have had only upper bounds. In practice, lower bounds are often more useful as they show that we need at least a certain number to guarantee the robber's capture. One such bound is due to Aigner and Fromme [1].

Theorem 3.7 [1] *If G is a graph which has no C_3 or C_4 subgraph, then $c(G) \geq \delta(G)$.*

Proof. Let $\delta(G) = d$ and suppose that $\delta(G) - 1$ cops play the game. Let $C = \{v_1, \dots, v_{d-1}\}$. Since $\delta(G) \geq d - 1$, there is a $w \in V(G) - C$. Suppose that the degree of w is ℓ . Thus, we can assume that the neighborhood of w is $\{v_1, \dots, v_k, w_1, \dots, w_{\ell-k}\}$, where $w_i \notin C$. By definition $\ell \geq d$ and $k \leq d - 1$. This implies that $\ell - k \geq 1$. Note that w_i and w_j have w as a common neighbor. If $w_i w_j \in E(G)$, then G has a C_3 subgraph. Further, if w_i and w_j have a common neighbor other than w , then G has a C_4 subgraph. Since G has no C_3 or C_4 subgraph, it follows that $N[w_i] \cap N[w_j] = \{w\}$ for all

$1 \leq i < j \leq \ell - k$. If C was a dominating set for G , then each of the $N(w_i)$ would have to contain at least one v_j , where $j \geq k + 1$. By the above argument, this would give us at least $\ell - k$ vertices. Taking these vertices together with v_1, \dots, v_k would give us at least $\ell \geq d$ total v_i . However, this contradicts there being only $d - 1$ v_i .

Suppose that the cop takes vertices $\{c_1, \dots, c_{d-1}\}$ initially. By the previous paragraph, the robber is able to place himself on a vertex r that is not equal to nor adjacent to any of the c_i . Since G has no C_3 or C_4 subgraph, after every move of the cop, at most $d - 1$ neighbors of R are occupied by cops or adjacent to them. Hence, the robber is able to move to the free neighbor. Thus, at least $\delta(G)$ cops are necessary to capture the robber. ■

To illustrate the utility of Theorem 3.7, we consider the Petersen graph shown in Figure 4.

Example 3.8 *The cop-number of the Petersen graph is $c(G) = 3$.*

Solution. Note that the Petersen graph G has no C_3 or C_4 subgraph and has $\delta(G) = 3$. By Theorem 3.7, $c(G) \geq \delta(G) = 3$. We now show that three cops are sufficient to catch the robber on the Petersen graph. Suppose that the cop takes vertices c , d , and f initially. The robber must take either g or j . Without loss of generality, suppose that $R = g$. The cop makes the following moves: $c \rightarrow b$, $d \rightarrow e$, and $f \rightarrow i$. The cop now controls two of the neighbors of g . Hence, the robber is forced to move to the remaining neighbor j . However, the cop controls a neighbor of j , hence they will capture the robber on the next turn. □

We now give several results regarding the cop number for Cartesian products of graphs.

Theorem 3.9 [25] *Let G and H be connected graphs. The cop number for the Cartesian product $G \square H$ satisfies*

$$c(G \square H) \leq c(G) + c(H).$$

Proof. Let $V(G) = \{x_1, \dots, x_p\}$ and $V(H) = \{y_1, \dots, y_q\}$. Let G_h denote the copy of G induced by $\{(g_1, h), \dots, (g_p, h)\}$. Similarly, let H_g be the copy of H induced by $\{(g, h_1), \dots, (g, h_q)\}$. Suppose that $c(G) = m$ and $c(H) = n$. Moreover, the (not necessarily distinct) vertices x_{i_1}, \dots, x_{i_m} are the starting position of the cops for their winning strategy on G . Likewise, the (not

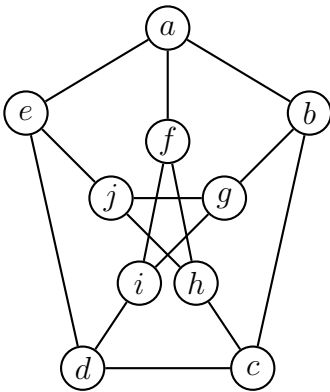


Figure 4: The Petersen graph

necessarily distinct) vertices y_{j_1}, \dots, y_{j_q} denote the starting position of the cops for their winning strategy on H . We want to show that $m + n$ cops are sufficient to capture the robber on the graph $G \square H$.

The cop begins by placing n cops on the vertices $(x_{i_1}, y_{j_1}), (x_{i_1}, y_{j_2}), \dots, (x_{i_1}, y_{j_n})$ on the copy $H_{x_{i_1}}$ and begins his winning strategy for H on this copy. Likewise, the cop places one additional cop on each copy of H . This allows him to catch at least the second coordinate of the robber on $G \square H$. If the robber is not in $H_{x_{i_1}}$, then from this moment forward the cop stays in $H_{x_{i_1}}$. This cop “shadows” the robber by going to the vertex (x_{i_1}, y_r) whenever the robber goes to vertex (x_i, y_r) .

The cop repeats this procedure on the copies $H_{x_{i_2}}, H_{x_{i_3}}, \dots, H_{x_{i_m}}$. At this point, the cop has achieved a position in which he has m cops placed in the vertices $(x_{i_1}, y_r), (x_{i_2}, y_r), \dots, (x_{i_m}, y_r)$. Meanwhile, the robber is in a vertex (x_i, y_r) . Denote the m cops in the vertices $(x_{i_1}, y_r), (x_{i_2}, y_r), \dots, (x_{i_m}, y_r)$ by C_1, \dots, C_m . Denote the remaining n cops by C'_1, \dots, C'_n .

From now on, whenever the robber’s move changes the second coordinate of the vertex, so will the cops C_1, \dots, C_m . These cops will always change their second coordinate in such a way that they all stay in the same copy G_{y_t} of G as the robber. If the robber in G_{y_r} changes his first coordinate, the cop instead moves the cops C_1, \dots, C_m according to the winning strategy for G . Notice that once placed, the cops C_1, \dots, C_m and robber *always* have the same second coordinate.

If the robber stays in a single copy H_{x_i} of H long enough, then the cops C'_1, \dots, C'_n will move into H_{x_i} by continually changing their first coordinate to match that of the robber. At this point, they will either catch the robber on

H_{x_i} by applying the winning strategy of H to this copy. Otherwise, they will force him to move to another copy of H . In other words, they will force him to make a move on G . In either case, $m + n$ cops are sufficient to capture the robber on $G \square H$. ■

Maamoun and Meyniel [15] gave a result for the Cartesian product of trees.

Theorem 3.10 [15] *Let T_1, \dots, T_k be trees. The cop number for the Cartesian product is given by*

$$c(\square_{i=1}^k T_i) = \left\lceil \frac{k+1}{2} \right\rceil.$$

A similar result about cycles was given by Neufeld and Nowakowski [16].

Theorem 3.11 [16] *Let C_1, \dots, C_k be cycles of length at least 4. The cop number for the Cartesian product is given by*

$$c(\square_{i=1}^k C_i) = k + 1.$$

They also proved a result regarding the Cartesian product of a combination of trees and cycles.

Theorem 3.12 [16] *Let C_1, \dots, C_k be cycles and $G = \square_{i=1}^k C_i$. Let T_1, \dots, T_j be trees and $H = \square_{i=1}^j T_i$. The cop number for the Cartesian product is given by*

$$c(G \square H) = k + \left\lceil \frac{j+1}{2} \right\rceil.$$

One of the most important open problems in this area is known as *Meyniel's Conjecture*. This conjecture first appeared in a paper by Frankl [9] based on a personal communication with Meyniel.

Conjecture 3.13 (Meyniel's Conjecture [9]) *Let G be a connected graph on n vertices. If n is sufficiently large, there is a constant $d > 0$ such that $c(G) \leq d\sqrt{n}$.*

There have been several attempts to prove Meyniel's Conjecture. Most of these attempts either prove an different upper bound or show that Meyniel's Conjecture holds for a particular class of graphs. For years, the best upper bound was given in Frankl's paper.

Theorem 3.14 [9] *Let G be a connected graph on n vertices. If n is sufficiently large, there is a constant $d > 0$ such that*

$$c(G) \leq d \frac{n \log(\log n)}{\log(n)}.$$

An improved bound was given by Chiniforooshan in 2008 [6].

Theorem 3.15 [6] *Let G be a connected graph on n vertices. If n is sufficiently large, there is a constant $d > 0$ such that*

$$c(G) \leq d \frac{n}{\log(n)}.$$

One of the results about graph classes is given below.

Theorem 3.16 [14] *If G is a graph on n vertices with diameter 2, then*

$$c(G) \leq 2\sqrt{n} - 1.$$

References

- [1] M. Aigner and M. Fromme. A game of cops and robbers. *Discrete Appl. Math.*, 8(1):1–11, 1984.
- [2] Brian Alspach. Searching and sweeping graphs: a brief survey. *Matematiche (Catania)*, 59(1-2):5–37 (2006), 2004.
- [3] József Beck. *Combinatorial games*, volume 114 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2008. Tic-tac-toe theory.
- [4] Alessandro Berarducci and Benedetto Intrigila. On the cop number of a graph. *Adv. in Appl. Math.*, 14(4):389–403, 1993.
- [5] Anthony Bonato and Richard J. Nowakowski. *The game of cops and robbers on graphs*, volume 61 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2011.
- [6] Ehsan Chiniforooshan. A better bound for the cop number of general graphs. *J. Graph Theory*, 58(1):45–48, 2008.

- [7] Fedor V. Fomin and Dimitrios M. Thilikos. An annotated bibliography on guaranteed graph searching. *Theoret. Comput. Sci.*, 399(3):236–245, 2008.
- [8] Aviezri S. Fraenkel. Combinatorial games: selected bibliography with a succinct gourmet introduction. *Electron. J. Combin.*, 1:Dynamic Survey 2, 45 pp. (electronic), 1994.
- [9] Peter Frankl. Cops and robbers in graphs with large girth and Cayley graphs. *Discrete Appl. Math.*, 17(3):301–305, 1987.
- [10] Shmuel Gal. *Search games*, volume 149 of *Mathematics in Science and Engineering*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.
- [11] Geňa Hahn. Cops, robbers and graphs. *Tatra Mt. Math. Publ.*, 36:163–176, 2007.
- [12] Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater. *Fundamentals of domination in graphs*, volume 208 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1998.
- [13] Kazimierz Kuratowski. Sur le problème des courbes gauches en topologie. *Fund. Math.*, 15:271283, 1930.
- [14] Linyuan Lu and Xing Peng. On Meyniel’s conjecture of the cop number. *J. Graph Theory*, 71(2):192–205, 2012.
- [15] M. Maamoun and H. Meyniel. On a game of policemen and robber. *Discrete Appl. Math.*, 17(3):307–309, 1987.
- [16] S. Neufeld and R. Nowakowski. A game of cops and robbers played on products of graphs. *Discrete Math.*, 186(1-3):253–268, 1998.
- [17] Richard Nowakowski and Peter Winkler. Vertex-to-vertex pursuit in a graph. *Discrete Math.*, 43(2-3):235–239, 1983.
- [18] T. D. Parsons. Pursuit-evasion in a graph. In *Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976)*, pages 426–441. Lecture Notes in Math., Vol. 642. Springer, Berlin, 1978.

- [19] T. D. Parsons. The search number of a connected graph. In *Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1978)*, Congress. Numer., XXI, pages 549–554. Utilitas Math., Winnipeg, Man., 1978.
- [20] Alain Quilliot. *Jeux et Points Fixes sur les graphes*. Thèse de 3ème cycle, Université de Paris VI, 1978.
- [21] Alain Quilliot. Discrete pursuit game. *Congr. Numer.*, 38:227–241, 1983.
- [22] Alain Quilliot. *Problèmes des jeux, de point, fixe, de connectivité et de représentation sur des graphes, des ensembles ordonnés et hypergraphes*. Thèse d’Etat, Université de Paris VI, 1983.
- [23] Aaron N. Siegel. *Combinatorial game theory*, volume 146 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013.
- [24] Cedric A. B. Smith. Graphs and composite games. *J. Combinatorial Theory*, 1:51–81, 1966.
- [25] Ratko Tošić. On cops and robber game. *Studia Sci. Math. Hungar.*, 23(1-2):225–229, 1988.
- [26] John von Neumann and Oskar Morgenstern. *Theory of games and economic behavior*. Princeton University Press, Princeton, NJ, anniversary edition, 2007. With an introduction by Harold W. Kuhn and an afterword by Ariel Rubinstein.
- [27] David Wells. *Games and mathematics*. Cambridge University Press, Cambridge, 2012. Subtle connections.
- [28] J. D. Williams. *The compleat strategyst*. Dover Publications, Inc., New York, second edition, 1986. Being a primer on the theory of games of strategy.