Decompositions and Graceful Labelings (Supplemental Material for Intro to Graph Theory)

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1 Introduction

A graph decomposition is a particular problem in the field of combinatorial designs (see Wallis [22] for more information on design theory). A *graph decomposition* of a graph H is a partition of the edge set of H. In this case, the graph H is called the *host* for the decomposition. Most graph decomposition problems are concerned with the case where every part of the partition is isomorphic to a single graph G. In this case, we refer to the graph G as the *prototype* for the decomposition. Further, we refer to the parts of the partition as *blocks*.

As an example, consider the host graph Q_3 as shown in Figure 1. In this case, we want to find a decomposition of Q_3 into isomorphic copies of the path on three vertices, P_3. We label the vertices of P_3 as an ordered triple (a, b, c). where ab and bc are the edges of P_3. The required blocks in the decompositions are $(0, 2, 4)$, $(1, 0, 6)$, $(1, 3, 2)$, $(1, 7, 5)$, $(3, 5, 4)$, and $(4, 6, 7)$.

Usually, the goal of a combinatorial design is to determine whether the particular design is possible. The same is true for graph decompositions. Namely, given a host graph H and a prototype G, determine whether there exist a decomposition of H into isomorphic copies of G. Often, this is a difficult problem. In fact, there are entire books written on the subject (for

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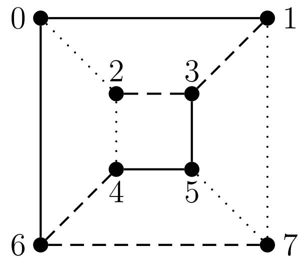

Figure 1: A P_3-decomposition of a Q_3

example, see Bosák [1] and Diestel [5]). Because of the difficulty of the problem, most researchers are only concerned with case where H is the complete graph on v vertices. This special case is often referred to as a *graphical design.* Note that we are using he variable v here because v is traditionally used to denote the number of treatments or *varieties* in a combinatorial design.

Even when we restrict our attention to the complete graph K_v (or more generally, λK_v where each edge of the complete graph has been replaced by an edge of multiplicity λ), progress in this area has often been slow. Results often proceed at a rate of one graph family at a time (sometimes only a single graph at a time). The survey articles written by Chee [3] and Heinrich [10] are a testament to the number of individual researchers who have contributed to this area of mathematics.

Because of the difficulty of the problem, we will be concerned with only a few elementary results.

2 Necessary conditions

As mentioned above, it is often difficult to determine whether a G-decomposition of H exists. However, there are a few elementary theorems that provide necessary conditions for the existence of a G-decomposition of H. Such theorems are important because if the graphs G and H violate these results, then we know immediately that there is no G-decomposition of H.

Theorem 2.1 *A necessary condition for the existence of a G-decomposition of H is that $|E(G)|$ divides $|E(H)|$.*

Proof. Suppose that such a decomposition exists. Then the edge set of H can be partitioned into b blocks of cardinality $|E(G)|$. Thus, $b|E(G)| = |E(H)|$.

Hence, $|E(G)|$ divides $|E(H)|$. ■

A natural question with any necessary condition is whether it is also sufficient. In other words, if $|E(G)|$ divides $|E(H)|$, then are we guaranteed the existence of a G-decomposition of H? As an example, consider $H = K_6$ and $G = K_3$. Note that $|E(K_6)| = 15$ and $|E(K_3)| = 3$. However, K_6 is a regular graph of degree 5. Whereas K_3 is regular of degree 2. So there is a problem when K_3-blocks meet at a single vertex of K_6. This idea is generalized in the following theorem.

Theorem 2.2 *Suppose that G is regular graph of degree r. A necessary condition for the existence of a G-decomposition of a graph H is that r divides $deg(v)$ for all $v \in V(H)$.*

Proof. Suppose that such a decomposition exists. In this decomposition, suppose that b_i G-blocks meet at $v_i \in V(H)$. Each of these blocks contain r of the edges incident with v_i. Thus, $deg(v_i) = rb_i$. Ergo, r must divide $deg(v)$ for all $v \in V(H)$. ■

In general, this is still not sufficient. Let H be the complete bipartite graph $K_{6,6}$. Take G to be the complete graph K_3. Note that $|E(K_{6,6})| = 36$ and $|E(K_3)| = 3$. So it does not violate Theorem **??**. Further, K_3 is regular of degree 2 and $K_{6,6}$ is regular of degree 6. Hence, Theorem 2.2 is not violated either. However, a graph is bipartite if and only if it does not contain any odd cycles. In particular, $K_{6,6}$ contains no K_3 subgraph.

3 Sufficiency

In the previous section, we discussed a couple of "easy" ways to show that decomposition fails to exist. In this section, we show that in some cases, it is "easy" to show that a decomposition exists. We will restrict our attention to the the case where our host graph H is the complete graph on v vertices, K_v. To facilitate our discussion, we will assume that the vertices of K_v are labeled with the integers modulo v, i.e., $\{0, 1, ..., v-1\}$. It is also useful to introduce a *modular absolute value* on the integers modulo v as follows:

$$|x|_v = \begin{cases} x & \text{if } 0 \le x \le v/2 \\ v - x & \text{if } v/2 < x < n. \end{cases}$$

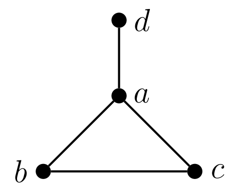

Figure 2: The graph G, $[a, b, c; d]$

We define the *length* of the edge between vertices i and j as $|i - j|_v$. Notice that if v is odd (say $v = 2t + 1$), then for all $k \in \{1, ..., t\}$, vertex $i \in K_v$ has exactly two edges of length k, namely $i + k \pmod{v}$ and $i - k \pmod{v}$. If v is even, then each i has an additional edge of length $i + \frac{v}{2}$. This extra edge can cause problems. We will see how to deal with this problem later.

Our goal will be to assign labels from $\mathbb{Z}_{2t+1}$ to the vertices of G such that each of the differences $\{1, ..., t\}$ appears exactly once on the edges of G. As an example of the methods of this section, we will assume that $H = K_v$ and that G is the graph obtained from K_3 by adding a pendant edge (see Figure 2).Note that Theorem **??** implies that a necessary condition for the existence of a G-decomposition of K_v is that $v \equiv 0 \pmod{8}$ or that $v \equiv 1 \pmod{8}$. We will denote the labels of a G-block by $[a, b, c; d]$ where a, b, and c are the labels on the vertices of the K_3 and d is the label of the pendant vertex adjacent to a.

For our example, consider the specific example where $v = 9$. Our set of differences is $\{1, 2, 3, 4\}$. Consider the labeling $[0, 1, 3; 4]$ on the vertices of G (this is sometimes called our *base block*). The edge 01 has length 1, the edge 13 has length 2, the edge 03 has length 3, and the edge 04 has length 4. To obtain our second block in the decomposition, we add 1 to each of the label of our base block to give us the block $[1, 2, 4; 5]$. Continue adding 1 to each of our labels, reducing modulo 9 when necessary (such a decomposition is often called *cyclic* because of this reduction). This gives us the required decomposition $[0, 1, 3; 4]$, $[1, 2, 4; 5]$, $[2, 3, 5; 6]$, $[3, 4, 6; 7]$, $[4, 5, 7; 8]$, $[5, 6, 8; 0]$, $[6, 7, 0; 1]$, $[7, 8, 1; 2]$, and $[8, 0, 2; 3]$. This set of blocks can be more compactly represented as $[i, i{+}1, i{+}3; i{+}4]$ for $i = 0, 1, ..., 8$. In this case, it is understood that our computations on the vertices are done modulo 9. Note that if $i = 0$, we get our original base block $[0, 1, 3; 4]$. For this reason, we often just list the base blocks for our decomposition.

As a second example, consider the case where $v = 17$. Our set of differences is $\{1,...,8\}$. We now need two base blocks to account for the four differences. Two such blocks are $[0,1,7;4]$ and $[0,2,5;8]$.

As mentioned earlier, the case where v is even (say $v = 2t$) often requires a slightly different strategy. In this case, we will label the vertices of K_v with the elements of $\mathbb{Z}_{2t-1} \cup \{\infty\}$ (the use of a point at infinity is borrowed from projective geometry). Thus, each of the vertices $0,1,...,2t-2$ have two edges of length k for $k = 1,...,t-1$ and one edge of length ∞. Thus, we want our base blocks to have each of these differences exactly once.

So if $v = 8$, then we can accomplish the required decomposition with the base block $[0,1,3;\infty]$. Likewise, we can accomplish the case where $v = 16$ with the base blocks $[0,1,7;\infty]$ and $[0,2,5;4]$. We generalize the above observations in the following theorem.

Theorem 3.1 *Let G be the graph obtained from K_3 by appending a pendant edge to one of the vertices. There exist a G-decomposition of K_v if and only if $v \equiv 0,1 \pmod 8$.*

One of the most famous graph decomposition problems is a K_3-decomposition of a K_v. This problem was solved independently by Kirkman [15] and Steiner [21]. Such a decomposition is often called a *Steiner triple system* in honor of Steiner. For more information on Steiner triple systems, see [4]. Our treatment of this subject will follow [13].

Theorem 3.2 *For $v \geq 3$ and $v \equiv 1,3 \pmod 6$, there exists a K_3-decomposition of K_v..*

Proof. Note that the case where $v \equiv 1 \pmod 6$, can further be divided into the subcases $v \equiv 1 \pmod{24}$, $v \equiv 7 \mod 24$, $v \equiv 13 \pmod{24}$, and $v \equiv 19 \pmod{24}$. Similarly, the case where $v \equiv 3 \pmod 6$ can be further divided into the subcases $v \equiv 3 \pmod{24}$, $v \equiv 9 \pmod{24}$, $v \equiv 15 \pmod{24}$, and $v \equiv 21 \pmod{24}$. We will consider each of these case in turn. In all cases, computations on the components are assumed to be done modulo v.

Suppose that $v \equiv 1 \pmod{24}$. Thus, there exists $k \in \mathbb{N}$ such that $v = 24k + 1$. Since $v = 1$ is not possible, we can assume that $k \geq 1$. For $j = 0,1,...,k-1$ and $i = 0,1,...,24k$ we use the blocks $[i, i+2j+1, i+j+11k+1]$, $[i, i+2j+3k+1, i+j+9k+1]$, $[i, i+2j+3k+2, i+j+6k+2]$, and $[i, i+2k, i+8k+1]$. If $k \geq 2$, then we also use the blocks $[i, i+2\ell+2, i+8k+\ell+2]$ for $\ell = 0,1,...,k-2$ and $i = 0,1,...,24k$.

Suppose that $v \equiv 3 \pmod{24}$. Thus, there exists $k \in \mathbb{N}$ such that $v = 24k+3$. Since the case where $v = 3$ is trivial, we can assume that $k \geq 1$. For $\ell = 0, 1, ..., 8k$, we use the blocks $[\ell, \ell+8k+1, \ell+16k+2]$. In addition, we use the blocks $[i, i+2j+1, i+j+11k+2]$, $[i, i+2j+2, i+j+8k+2]$, $[i, i+2j+3k+2, i+j+9k+2]$, and $[i, i+2j+3k+1, i+j+6k+1]$ for $j = 0, 1, ..., k-1$ and $i = 0, 1, ..., 24k+2$.

Suppose that $v \equiv 7 \pmod{24}$. Thus, there exists $k \in \mathbb{N}$ such that $v = 24k+7$. We use the blocks $[i, i+2k+1, i+8k+3]$ for $i = 0, 1, ..., 24k+6$. If $k \geq 1$, then we use the additional blocks $[i, i+2j+1, i+j+11k+4]$, $[i, i+2j+2, i+j+8k+4]$, $[i, i+2j+3k+3, i+j+9k+4]$, and $[i, i+2j+3k+2, i+j+6k+3]$ for $j = 0, 1, ..., k-1$ and $i = 0, 1, ..., 24+6$.

For $v = 9$, we suppose that the vertex set of K_9 is $\{0, 1, 2, 3, 4, 5, 6, 7, \infty\}$. Thus, the set of differences is $\{1, 2, 3, 4, \infty\}$. It is important to note that with the difference 4, we can only rotate the corresponding block half way through the treatments. The reason for this is that the difference 4 is its own additive inverse modulo 8. Thus, if we do a complete rotation of a block, say $[0, 1, 4]$, then this would result in the block $[4, 5, 0]$. Hence, the pair of treatments 0 and 4 would be repeated. Thus, we partition our differences into the sets $\{1, 2, 3\}$ (similar to the previous examples) and $\{4, \infty\}$. The first partition generates the blocks $[i, i+1, i+3]$ for $i = 0, 1, ..., 7$. The second partition generates the blocks $[j, j+4, \infty]$ for $j = 0, 1, 2, 3$. Direct inspection confirms that these blocks give the required decomposition.

Suppose that $v \equiv 9 \pmod{24}$. Thus, there exists $k \in \mathbb{N}$ such that $v = 24k+9$. The case where $v = 9$ is done above. Hence, we can assume that $k \geq 1$. For $\ell = 0, 1, ..., 8k+2$, use the blocks $[\ell, \ell+8k+3, \ell+16k+6]$. For $k \geq 1$, we use the additional blocks $[i, i+2k-1, i+5k+2]$, $[i, i+3k, i+12k+3]$, $[i, i+3k+1, i+12k+5]$, $[i, i+2j+3k+2, i+j+9k+5]$, and $[i, i+2j+3k+5, i+j+6k+4]$ for $j = 0, 1, ..., k-1$ and $i = 0, 1, ..., 24k+8$. If $k \geq 2$, then we additionally use the blocks $[i, i+2j+1, i+j+11k+4]$ and $[i, i+2j+2, i+j+8k+4]$ for $j = 0, 1, ..., k-2$ and $i = 0, 1, ..., 24k+8$.

Suppose that $v \equiv 13 \pmod{24}$. Thus, there exists $k \in \mathbb{N}$ such that $v = 24k+13$. If $k \geq 0$, then we use the blocks $[i, i+2k+1, i+8k+4]$ and $[i, i+3k+2, i+12k+7]$ for $i = 0, 1, ..., 24k+12$. If $k \geq 1$, then we use the additional blocks $[i, i+2j+1, i+j+11k+6]$, $[i, i+2j+2, i+j+8k+5]$, $[i, i+2j+3k+4, i+j+9k+6]$, and $[i, i+2j+3k+3, i+j+6k+4]$ for $j = 0, 1, ..., k-1$ and $i = 0, 1, ..., 24k+12$.

Suppose that $v \equiv 15 \pmod{24}$. Thus, there exists $k \in \mathbb{N}$ such that $v = 24k+15$. For $\ell = 0, 1, ..., 8k+4$, use the blocks $[\ell, \ell+8k+5, \ell+$

$16k+10]$. Additionally, we use the blocks $[i, i+3k+2, i+12k+8]$ and $[i, i+2j+3k+3, i+j+6k+4]$ for $j=0,1,...,k$ and $i=0,1,...,24k+14$. If $k \geq 1$, then we use the additional blocks $[i, i+2j+1, i+j+11k+7]$, $[i, i+2j+2, i+j+8k+6]$, and $[i, i+2j+3k+4, i+j+9k+7]$ for $j=0,1,...,k-1$ and $i=0,1,...,24k+14$.

Suppose that $v \equiv 19 \pmod{24}$. Thus, there exists $k \in \mathbb{N}$ such that $v = 24k+19$. If $k \geq 0$, then we use the blocks $[i, i+2k+1, i+8k+5]$, $[i, i+3k+2, i+12k+8]$, and $[i, i+3k+3, i+12k+10]$ for $i=0,1,...,24k+18$. If $k \geq 1$, then we use the additional blocks $[i, i+2j+1, i+j+11k+8]$, $[i, i+2j+2, i+j+8k+6]$, $[i, i+2j+3k+5, i+j+9k+8]$, and $[i, i+2j+3k+4, i+j+6k+5]$ for $j=0,1,...,k-1$ and $i=0,1,...,24k+18$.

Suppose that $v \equiv 21 \pmod{24}$. Thus, there exists $k \in \mathbb{N}$ such that $v = 24k+21$. For $\ell = 0,1,...,8k+6$, use the blocks $[\ell, \ell+8k+7, \ell+16k+14]$. Additionally, use the blocks $[i, i+2j+1, i+j+11k+10]$, $[i, i+2j+3k+3, i+j+9k+8]$, and $[i, i+2j+3k+4, i+j+6k+6]$ for $j=0,1,...,k$ and $i=0,1,...,24k+20$. If $k \geq 1$, then use the additional blocks $[i, i+2j+2, i+j+8k+8]$ for $j=0,1,...,k-1$ and $i=0,1,...,24k+20$. ■

One of the most famous open problems in graph decompositions is that of Ringel's Conjecture [19].

Conjecture 3.3 (Ringel's Conjecture [19]) *If T is a tree with q edges, there exists a T-decomposition of K_{2q+1}.*

4 Graceful Labelings

In an effort to solve Ringel's Conjecture, Rosa introduced several methods of labeling the vertices of graphs in order to achieve a base block as described above [20]. The most important of these labelings was popularized by Golomb [9] under the name of *graceful labelings*. The dynamic survey by Gallian [8] and its over 2100 references is a testament to the amount of research done on graceful and related labelings.

Definition 4.1 *Let G be a graph with q edges. A* graceful labeling *on G is an injective function $f : V(G) \to \{0,1,...,q\}$ such that $\{|f(x)-f(y)| : xy \in E(G)\} = \{1,...,q\}$. A graph is* graceful *if it has a graceful labeling.*

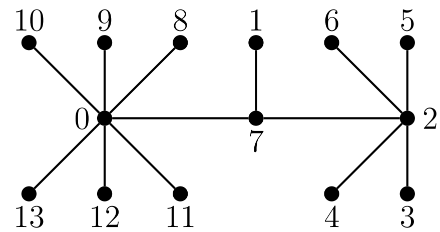

Figure 3: A gracefully labeled caterpillar - $P_3(6,1,4)$

In our next result, we show that it a graph G on q edges has a graceful labeling, then there exists a G decomposition of K_{2q+1}.

Theorem 4.2 *Let G be a graph with q edges. If G is graceful, then there exists a G-decomposition of K_{2q+1}.*

Proof. From a graceful labeling of G, we define copies of G in K_{2q+1}. These copies are G_0, G_1,..., G_{2q}. The vertices of G_k are k, $k+1$,...,$k+q$ (mod $2q+1$), where $k+i$ is adjacent to $k+j$ in G_k if and only if i is adjacent to j in our gracefully labeled base block. Thus each of the G_k has exactly one of the differences 1,...,q. Further the edge between $k+i$ and $k+j$ in K_{2q+1} is covered by the corresponding edge in G_k. Hence, the graceful labeling induces a decomposition of K_{2q+1}. ■

We now give some examples of graceful labelings on graphs. Recall that a caterpillar can be obtained from the path on the vertices x_1,...,x_n by appending a_i pendants $x_{i,1}$,...,x_{i,a_i} to x_i. Such a caterpillar is denoted $P_n(a_1,...,a_n)$ (see Figure 3). Note that the set of caterpillars include all stars, double stars, and paths.

Theorem 4.3 *All caterpillars are graceful.*

Proof. Note that the caterpillar $P_n(a_1,...,a_n)$ has $a_1+\cdots+a_n+n$ vertices and $q=a_1+\cdots+a_n+n-1$ edges. It suffices to give the required labeling.

Label $f(x_1)=0$, $f(x_{1,i})=q-i+1$ for $i=1,...,a_1$, and $f(x_2)=q-a_1$. This gives us edge labels $q-a_1$, $q-a_1+1$,...,q. Now, label the pendants of x_2 with $f(x_{2,i})=i$ for $i=1,...,a_2$. We also label $f(x_3)=a_2+1$. This gives us the edge labels $q-a_1-a_2-1$, $q-a_1-a_2$,...,$q-a_1-1$. We continue this process alternating high and low labels on the vertices to give the required labeling. ■

Of course, not all trees are caterpillars. In Figure 4, we give all a graceful labeling on all non-caterpillar trees with nine vertices or less.

The above information leads credence to the following conjecture.

Conjecture 4.4 (Graceful Tree Conjecture [20]) *All trees are graceful.*

If the Graceful Tree Conjecture were true, then Theorem 4.2 would imply Ringel's Conjecture. There has been a great deal of work finding graceful labelings on graphs and making partial progress on the Graceful Tree Conjecture. A short collection of these results are given below.

Proposition 4.5 *The following graphs are known to have a graceful labelling:*

(i) Caterpillars [20].

(ii) Trees with at most four endpoints [12, 14, 20, 24].

(iii) Trees of diameter at most five [11].

(iv) Trees with at most 35 vertices [6].

(v) Complete bipartite graphs [9, 20].

(vi) Cycles of length n where $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$ [16].

(vii) Gear graphs, the Petersen graph, and polyhedral graphs [23].

(viii) $C_n \vee K_1$ (i.e., Wheel Graphs) [7].

(ix) The n-dimensional hypercube, Q_n [17, 18].

(x) The graph obtained by subdividing each edge of a graceful tree [2].

One of the advantages to graceful labelings (as opposed to the more general labelings described in Rosa) is that it is possible to show that certain graphs do not have a graceful labeling. One such result is given below.

Theorem 4.6 *Suppose that G is an eulerian graph with q edges. If G is graceful, then $q \equiv 0, 3 \pmod 4$.*

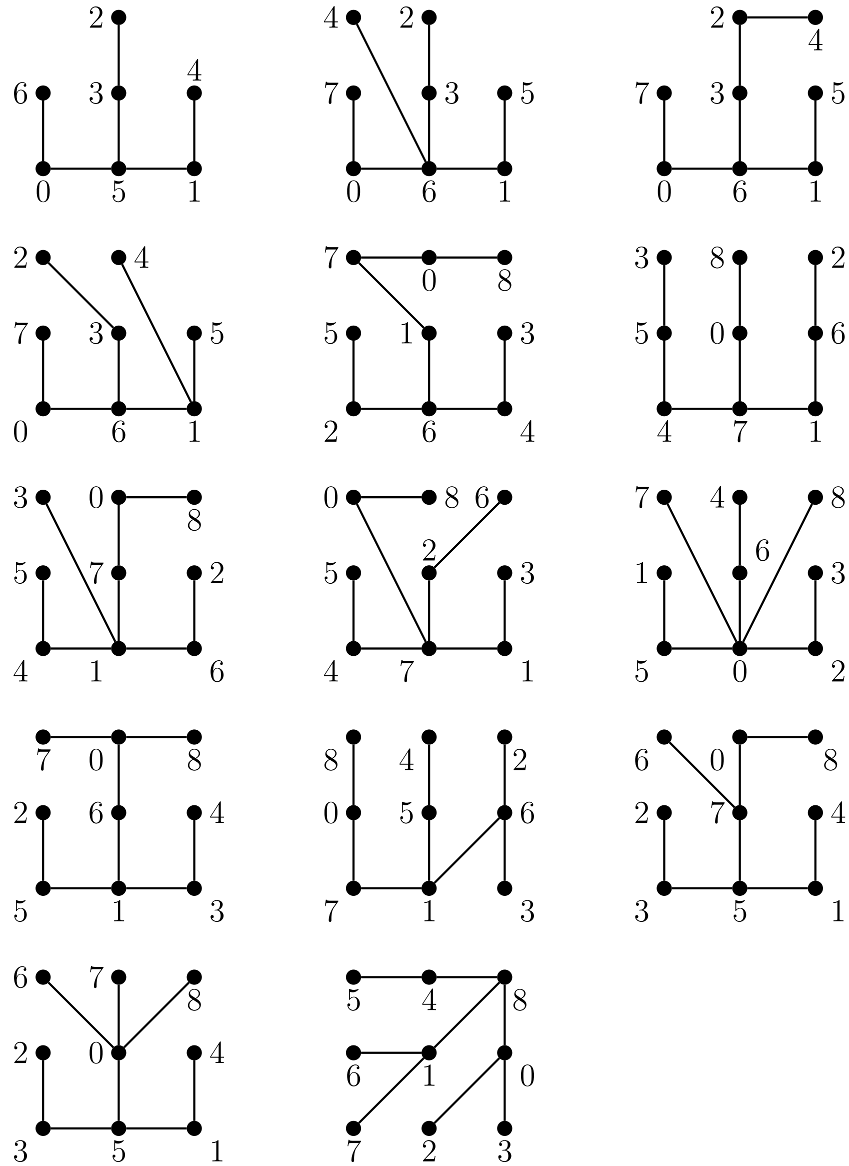

Figure 4: Graceful labelings for non-caterpillar trees with $n(G) \leq 9$

Proof. Recall that a (connected) graph is eulerian if and only if every vertex is of even degree.

Suppose that f is a graceful labeling on G. Suppose that $E(G) = \{z_1, ..., z_q\}$ and the endpoints of z_i are the vertices x_i and y_i. Without loss of generality, we assume that $f(x_i) > f(y_1)$ for all $i = 1, ..., q$. Thus the edge label of z_i is $f'(z_i) = f(x_i) - f(y_i)$. Thus,

$$\sum_{i=1}^{q} f'(z_i) = \sum_{i=1}^{q} f(x_i) - \sum_{i=1}^{q} f(y_i)$$

$$= \sum_{i=1}^{q} f(x_i) + \sum_{i=1}^{q} f(y_i) - 2\sum_{i=1}^{q} f(y_i).$$

Consider the list x_1,...,x_q, y_1,...,y_q. The number of times each vertex appears is equal to its degree. Since G is eulerian, each vertex is of ven degree. This implies that

$$\sum_{i=1}^{q} f(x_i) + \sum_{i=1}^{q} f(y_i) \quad \text{is even.}$$

Therefore,

$$\sum_{i=1}^{q} f'(z_i) \quad \text{is even.}$$

Say

$$\sum_{i=1}^{q} f'(z_i) = 2k, k \in \mathbb{Z}.$$

Since f is graceful, the edge labels are distinct elements of the set $\{1, ..., q\}$. Hence,

$$\sum_{i=1}^{q} f'(z_i) = 1 + \cdots + q = \frac{q(q+1)}{2}$$

$$\Rightarrow 2k = \frac{q(q+1)}{2}$$

$$\Rightarrow k = \frac{q(q+1)}{2} \in \mathbb{Z}.$$

Thus, $4|q$ or $4|q+1$. Equivalently, $q \equiv 0, 3 \pmod 4$. ∎

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