e, The Younger, Less Popular, Cousin of π

Robert A. Beeler, Ph.D.

East Tennessee State University

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References and Acknowledgements

This is a very brief history of the number e. This talk is based on the project that I did for my Master's degree at the University of Tennessee, Knoxville under the direction of Dr. William Wade. The primary reference that I used for the project was the book *e: The Story of a Number* by Eli Maor.

What is e?

The number e is the base of the natural exponential function. This is the only non-trivial function whose derivative is itself. Defined as a limit:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n .$$

As an infinite (MacLaurin) series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

And the first few digits of e:

$$e \approx 2.71828182845904523536028747135266249775724709369995\ldots$$

Prehistory - John Napier

In 1593, Scottish nobleman John Napier (1550-1617) published an attack on the Roman Catholic Church, *The Plaine Discovery of the Whole Revelation of Saint John*. This book claimed that the Pope was the Anti-Christ and the world would end between the years 1688 and 1700 (it didn't). Napier believed that his future reputation would be dependent on this manifesto. Fortunately, this book is almost completely forgotten. We instead remember Napier for his role in the development of the logarithm.

Prehistory - Astronomy

As an educated man, Napier was no doubt familiar with the great astronomical discoveries of his time:

- 1543 - Copernicus publishes *De revolutionibus orbium coelestium*.
- 1573 - Tycho Brahe publishes *De nova stella*.
- 1609 - Keplar publishes his laws of planetary motion in *Astronomia nova*.
- pre-1609 - Galileo builds a refracting telescope.

As these breakthroughs took place, scientists needed an efficient means of doing their calculations.

Prehistory - Logarithms

Traditionally, it has always been easier to add or subtract rather than multiply or divide. Napier was no doubt familiar with trigonometric identities such as

$$\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y)).$$

Formulas such as this reduced the operation of multiplication (which is hard) to addition (which is less hard).

Prehistory - Logarithms

In Napier's time, trigonometric tables were accurate to 10^{-7}. As a result, Napier used ten-millionth as part of his "base." For this to be useful, he made his base close to one by taking his base to be $1 - .1^7$. Napier also wanted to avoid fractions and decimals, so he multiplied by 10^7. So if

$$N = 10^7(1 - 10^{-7})^L,$$

then L is the Naperian logarithm of N.

Prehistory - Logarithms

Napier spent twenty years of his life computing tables of logarithms. In 1614, these were published in *Mirifici Logarithmorum Canonis Descriptio*. This brought the attention of Henry Briggs (1561-1630) who met with Napier in 1615. Briggs suggested the use of the "common base" of 10. Briggs published a table of common logarithms in *Arithmetica Logarithmica* in 1624.

The Discovery - Jakob Bernoulli

Jakob Bernoulli (1654-1705) studied compound interest in 1683. Consider the compound interest formula:

$$P\left(1+\frac{r}{n}\right)^{nt},$$

where P is the principal, r is the rate, t is time. and n is compounding frequency.

The Discovery - Jakob Bernoulli

Suppose that we borrow one dollar for a period of one year at one hundred percent interest from a loan shark. However, the loan shark decides how often they compound the interest. How bad can the following formula get:

$$\left(1 + \frac{1}{n}\right)^n ?$$

The Discovery - Jakob Bernoulli

$$\left(1+\frac{1}{1}\right)^{1} = 2, \quad \left(1+\frac{1}{2}\right)^{2} = 2.25, \quad \left(1+\frac{1}{6}\right)^{6} \approx 2.5216,$$

$$\left(1+\frac{1}{12}\right)^{12} \approx 2.613 \ \left(1+\frac{1}{52}\right)^{52} \approx 2.6926 \ \left(1+\frac{1}{365}\right)^{365} \approx 2.7146$$

$$\left(1+\frac{1}{1200}\right)^{1200} \approx 2.717, \quad \left(1+\frac{1}{12000000}\right)^{12000000} \approx 2.718280587,$$

$$\left(1+\frac{1}{32000000}\right)^{32000000} \approx 2.71828178599$$

The limit as $n \to \infty$ seems to exist.

The Discovery - Jakob Bernoulli

Bernoulli showed that the sequence $\left(1+\frac{1}{n}\right)^n$ is bounded above and increasing. So by the monotonic convergence principle, the limit exists. In our modern notation, we call this limit e. Jakob Bernoulli (along with his brother Johann Bernoulli) were among the first to discover the derivative rule:

$$\frac{d}{dx}e^x = e^x.$$

It is because of this property that Rudin describes the function as "undoubtedly the most important function in mathematics".

Rise to Prominence - Leonard Euler

Leonard Euler (1707-1783) did much to elevate the exponential to the the prominence it has today. In 1748, Euler's widely circulated textbook *Introductio in analysin infinitorum* introduced and popularized the modern notations for e, π, and i. In particular, he gave the exponential function and the logarithmic functions separate and independent definitions:

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n, \ln x = \lim_{n\to\infty} n\left(x^{1/n} - 1\right).$$

Rise to Prominence - Leonard Euler

Euler proved a number of theorems that included e. For example, he proved that e and e^2 are irrational. However, in 1749 he also discovered one of the many formula that bear his name:

$$e^{ix} = cosx + i \sin x.$$

Note that there are a couple of fun things we can do with this:

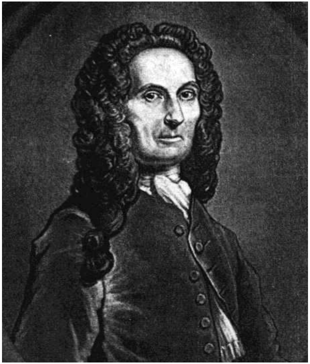
$$x = \pi \Rightarrow e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0.$$

$$x = \frac{\pi}{2} \Rightarrow e^{i\pi/2} = i \Rightarrow e^{-\pi/2} = i^i.$$

Other Work - de Moivre

In 1733, Abraham de Moivre (1667-1754) published *Approximatio ad summan terminorum binomi* $(a+b)^n$ *in seriem expansi* which included another well-known formula that includes both e and π:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$



Stirling's Approximation

In 1733, de Moivre also discovered the popular approximation for $n!$:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n .$$

Yet another formula that has both e and π. Today, this approximation is known as Stirling's approximation after James Stirling (1692-1770).

Transcendence - Liouville

Transcendental numbers are numbers that are not the root of a polynomial with integer coefficients. In 1844, Joseph Liouville (1809-1882) constructed an infinite class of transcendental numbers. However, these numbers are somewhat unsatisfying as they are little more than mathematical pathologies. An example of such a number is:

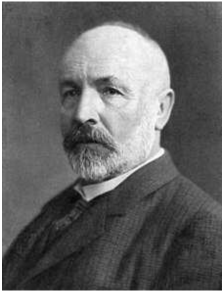
$$\sum_{n=0}^{\infty} \frac{1}{10^{n!}} = .1100010000000000000000010....$$

Transcendence - Hermite

In 1873, Charles Hermite (1822-1901) became the first person to prove that a given number is transcendental when he proved that e is transcendental.

Other Transcendental Results

In 1874, Georg Cantor (1845-1918) proved that there are in fact more transcendental numbers than anything else.



Other Transcendental Results (Part 2)

In 1882, Ferdinand von Lindemann (1852-1939) proved that if $\alpha \neq 0$, then at least one of e^{α} and α is transcendental. Taking $\alpha = i\pi$ shows that π is transcendental. Lindemann's proof is based heavily on the earlier proof of Hermite.

Other Transcendental Results (Part 3)

In 1885, Karl Weierstrass (1815-1897) proved a generalization of Lindemann's result that is commonly known as the Lindemann-Weierstrass Theorem. This theorem states that if $a_1, \ldots, a_n$ are non-zero algebraic numbers and $\alpha_1, \ldots, \alpha_n$ are distinct algebraic numbers, then

$$a_1 e^{\alpha_1} + \cdots + a_n e^{\alpha_n} \neq 0.$$

In particular, this means that if α is algebraic and $\alpha \neq 0$, then $sin(\alpha)$ and $cos(\alpha)$ are transcendental.

Other Transcendental Results (Part 4)

In 1934, Aleksandr Gelfond (1906-1968) and Theodor Schneider (1911-1988) independently proved that if a and b are both algebraic numbers with $a \notin \{0, 1\}$ and $b \notin \mathbb{Q}$, then a^b is transcendental (Hilbert's Seventh Problem). Taking $a = i$ and $b = -2i$ shows that e^π is transcendental.

Are you deranged?

In 1708, Pierre Rémond de Montmort (1678-1719) considered the combinatorial problem of *derangements*. Suppose that we have n people check their hats at the theatre. But ticket check girl loses all the tickets and starts handing out all the hats at random. How many ways can no one get their own hat back? In 1713, Montmort solved this problem in "Essay d'analyse sur les jeux de hazard":

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx \frac{n!}{e}.$$

Continued Fraction - Another Neat Formula

One final cute formula for e. If you express e as a continued fraction, it has a very neat pattern:

$$2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}}}}$$

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots, 1, 2n, 1, \ldots]$$

Open Problems

- Is e a normal number? In other words, are its digits (in any base) statistically random?
- Transcendence of $e\pi$, $e + \pi$, and π^e?
- Is e the root of any polynomial with coefficients in $\mathbb{Q}(\pi)$?

Questions?

"This talk has been brought to you by the number e."