Major MacMahon's Revenge

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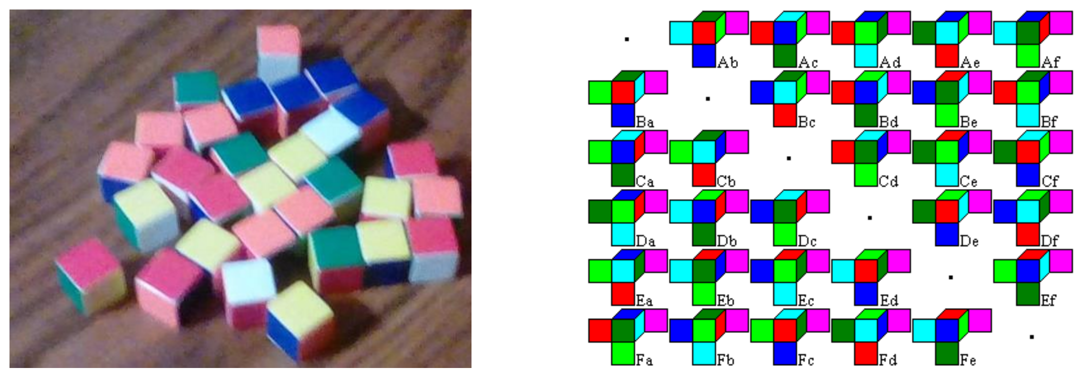
Introduction

Figure 1: Major Percy Alexander MacMahon (1854-1929)

Major Percy Alexander MacMahon (Figure 1) was an English mathematician who worked in the field of combinatorics. He is most famous for his work on the theory of partitions and combinatorial analysis. However, this article is intended to showcase puzzles described in, or inspired by, his book *New Mathematical Pastimes*, currently being reprinted by Forgotten Books.

I first discovered the puzzles of Major MacMahon while reading Ernö Rubik's article "In Play," published in *Rubik's Cubic Compendium*. In this article, Rubik describes MacMahon's cubes. Each cube has six faces and will be colored with one of six colors in such a way that no color is used on two faces of the same cube. Up to rotations of the cube, there are thirty distinct ways to color the cubes in this way (see Figure 1a). MacMahon's puzzle is to select one cube from the set of thirty. Then select eight of the remaining cubes to form a 2X2X2 cube so that the outer faces match those of the selected cube and when two cubes touch in the interior, the colors on the adjacent faces match. A complete solution to this problem was given by Conway. Conway's solution is to arrange the cubes into a 6X6 matrix as shown in Figure 1b. Suppose that we select cube Db as our prototype. We then remove its "mirror" Bd. The eight cubes needed for the solution are the remaining cubes that were in line with Bd, namely Ad, Cd, Ed, Fd, Ba, Bc, Be, and Bf. This gives all solution to MacMahon's 2X2X2puzzle. However, we can also consider similar puzzles, such as constructing a 2X2X3 prism, a 2X2X4 prism, a 2X3X3 prism, a 2X3X5 prism (suggested by MacMahon), or a 3X3X3 cube according to the above restrictions.



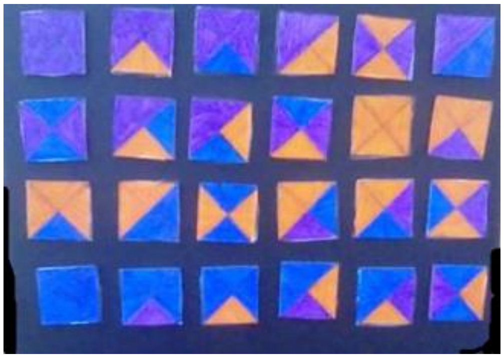
a) 30 distinctly colored cubes b) Conway's solution to the 2X2X2 problem

Figure 2: MacMahon's colored cubes

EDGE-MATCHING PUZZLES

MacMahon's puzzles consists of a collection of identical shapes that have been partitioned into regions. Throughout this article, we will assume that the regions are identically shaped. Each region is then assigned a color. The set of tiles is all distinct tiles that can be made with those colors. The usual goal is to arrange the full set of tiles into a single solid shape such that the following restrictions hold: (1) When two tiles touch on an edge, the colors on those edges match. (2) The border of the shape is a single color. Wade Philpott extended this to consider not only a single shape that satisfied the above restrictions, but to also consider the *duplication* problem (two identical shapes that satisfy the above restrictions) and the *triplication* problem (three identical shapes that satisfy the above restrictions). As MacMahon and others have observed, these puzzles can be thought of as generalized dominoes. In this article, we give several examples of MacMahon-type puzzles and their solutions. While this article only considers single player versions (solitaires), variations for two or more players are also available.

Perhaps the most famous of MacMahon's recreations is his 3-colored squares. The tile set is created connecting the opposite corners of squares, resulting in four equal quadrants. Each section is assigned one of three colors. The tile set consists of all possible colorings, up to rotation of the tiles. When m colors are used, there are $\frac{1}{4}(m^4 + m^2 + 2m)$ possible tiles. When $m = 3$, this results in 24 distinct tiles, as shown in Figure 3a. The most common goal is to create a 4X6 rectangle according to the above restrictions. According to Long, there are 13,328 solutions (up to rotating the board or permuting the colors). One such solution is given in Figure 3b. It is worth noting that it is impossible to construct solid rectangles with dimensions other than 4X6 according to the above restrictions. For example, it is impossible to construct a 2X12 rectangle as the border would require that all 24 tiles contain the same color, which is clearly not the case. It is slightly more difficult to argue that a 3X8 rectangle is likewise impossible. To see this, suppose that we want the border to be purple. While we have eighteen tiles with purple and eighteen tiles on the border, the "bowtie" tiles would require an extra purple tile in the center of the rectangle. Hence, the 3X8 rectangle is impossible. However, it is possible to achieve rectangular shapes that are not solid. For example, Figure 3c shows the tiles arranged into a 5X5 square with a hole in the center. In addition, we present possible solutions to the duplication and triplication problems in Figure 3d.



a) Pieces for MacMahon Squares

b) A 4X6 rectangular solution

c) A 5X5 square solution with a hole

c) Solutions to the duplication and triplication problems

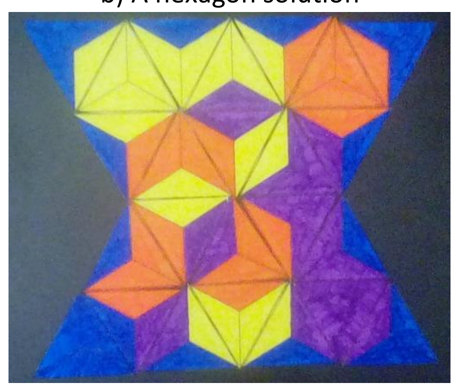
Figure 3: MacMahon Squares

MacMahon also proposed a set tiles in which equilateral triangles are divided into thirds by connecting the corners to the center. Each of the resulting sections is assigned a color and the tile set consists of all possible colorings, up to rotation. When m colors are used, there are $\frac{1}{3}(m^3 + 2m)$ distinct tiles. In particular, when $m = 3$ this results in 24 unique tiles as shown in Figure 4a. As usual, the goal is to assemble the tiles into a single solid figure, according to the above restrictions. Because each color appears on eighteen edges (an even number), it follows that any such polygon must have an even number of tiles along its perimeter. Further, because the solid color tiles cannot contribute all three of their edges to the perimeter, a polygon whose perimeter consists of eighteen or more tiles is impossible. Likewise, the smallest perimeter that can be achieved with 24 tiles is twelve. The only polygon that achieves this perimeter of twelve is the hexagon, as shown in Figure 4b. Philpott cataloged all polygons that achieve a perimeter of fourteen or sixteen (these are reproduced in Gardner's "Colored Triangles and Cubes"). An example of a polygon that achieves a perimeter of fourteen is the 3X4 parallelogram, which is shown in Figure 4c. An example of a polygon that achieves a perimeter of sixteen is the "hourglass" shown in Figure 4d. Solutions to the duplication and triplication problems are given in Figures 4e and 4f, respectively. Since the standard MacMahon triangles have four different colors, it is possible to create four congruent shapes that satisfy the above restrictions. One example is the four hexagons depicted in Figure 4g.

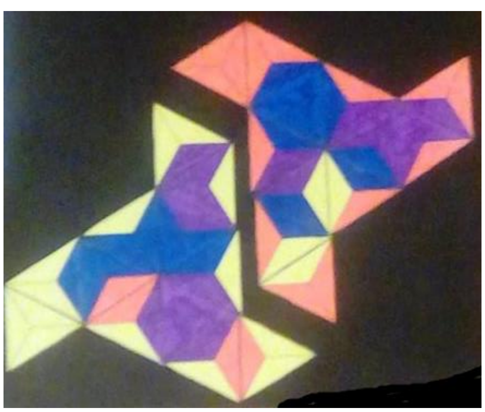
a) A set of MacMahon Triangles

b) A hexagon solution

c) A 3X4 parallelogram solution

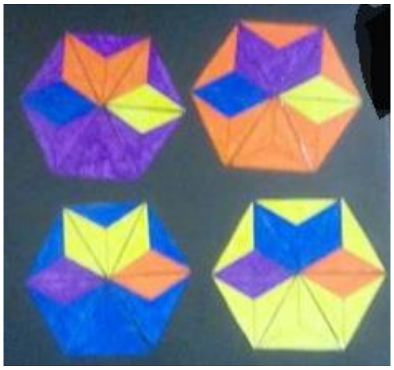


d) An “hourglass” solution



e) A solution to the duplication problem – two spirals

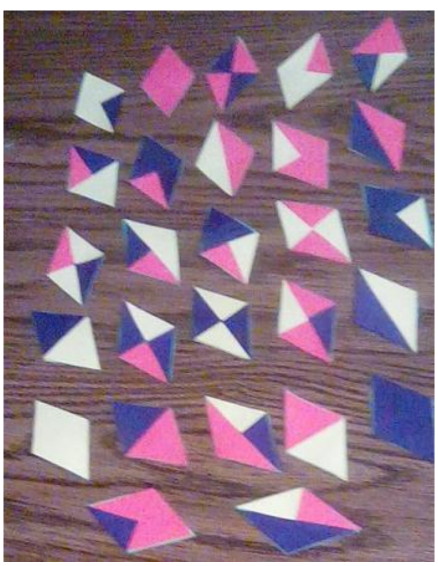
f) A solution to the triplication problem – three diamonds



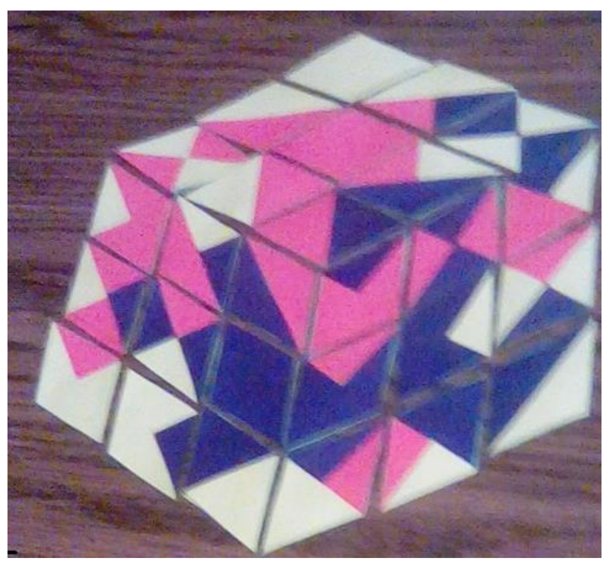
g) Four hexagons

Figure 4: MacMahon Triangles

A third variation is *Abaroth's Rhombi* invented by Dave Barlow in 2013. Abaroth's Rhombi consist of a set of non-square rhombi. Each rhombus is divided into fourths using the diagonals. Each quadrant is colored one of three colors. Unlike the other puzzles in this article, the tiles are two sided. Hence, we only consider distinct tiles up to 180 degree rotations and flips. With *m* colors, the number of distinct tiles is $\frac{1}{4}(m^4 + 3m^3)$. In particular, when $m = 3$, this gives twenty-seven distinct tiles, as shown in Figure 5a. As with the other puzzles in this section, the goal is to make a solid shape in which the border is a single color and if two tiles are adjacent, then they have the same color along their shared edge. A hexagonal solution that satisfies the above is given in Figure 5b. Kadon Enterprises produces a variation on this set called "The Rhombs of Multimatch-I". Their set includes forty-five tiles because they are one-sided. In other words, their tile set is all three colored rhombi up to rotation, but not reflection.



a) Abaroth's Rhombi pieces



b) A hexagon solution

Figure 5: Abaroth's Rhombi

EDGE-MATCHING PUZZLES WITH RELAXED BORDERS

In the previous section, we discussed puzzles in which adjacent tiles must share the same color on their shared edge and the border of the shape must be a single color. In this section, we look at three puzzles in which the border condition is relaxed. In other words, two adjacent tiles must still share the same color on their shared edge. However, the border of the resulting solid need not be (and in fact, it is impossible to be) a single color.

The first, is a variation considered by MacMahon in his *New Mathematical Pastimes*. As with MacMahon squares, each tile is divided into fourths using the diagonals. Each quadrant each colored one of five colors, under the restriction that no color appears twice on a single tile. This results in a set of 5*4*3*2/4=30 distinct tiles, up to rotation. The full set of thirty tiles is pictured in Figure 6a. However, MacMahon was only interested in puzzles in which there were at most twenty-four tiles. For this reason, MacMahon added the additional restriction that one color, say purple, must appear on every tile. This additional restriction reduces the number of tiles to twenty-four. Figure 6b illustrates one of MacMahon's solutions to the reduced puzzle. One of the remarkable things about MacMahon's solution is that while a single border color is impossible, the left and right borders are the same solid color. Edge matched solutions using the full set of tiles are possible, as shown in Figure 6c. One of the interesting things about this solution is that the top three rows are mirrored in the bottom three rows.

a) Pieces for MacMahon's 5-color variant

b) MacMahon's reduced solution

c) A solution with all 30 tiles

Figure 6: MacMahon's variant - 5-colored squares in which each color appears at most once on a square

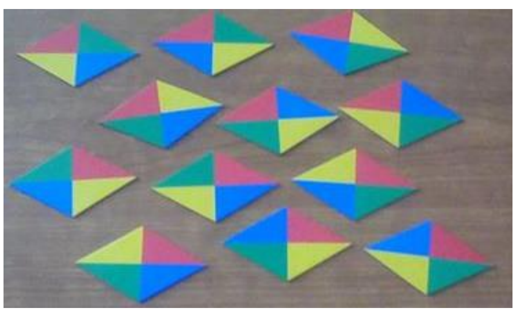
For our second variant in this section, we consider the variant known as *Marshall Squares*. Marshall Squares were invented by William Marshall in 1983. As with MacMahon Squares, each square tile is divided into quadrants using the diagonals. Each sector is colored one of five colors. However, if a color appears on a tile, it appears either two or four times. Up to rotation, this restriction gives five solid tiles, ten tiles in which each color appears twice on each half of the tile, and ten in which each color appears once on each half of the tile. The full set of twenty-five tiles are pictured in Figure 7a. As before, we desire to make a solid shape in which two adjacent tiles must have the same color along the square edge. A square solution that satisfies this is shown in Figure 7b.

a) Pieces for Marshall's squares

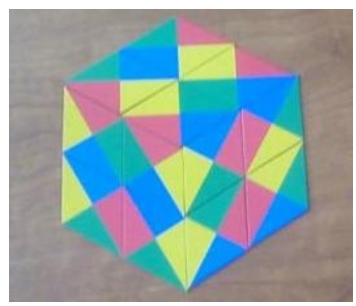
b) A square solution

Figure 7: Marshall's squares

IZZI 2 is a puzzle invented by Frank Nichols in 1992. The tiles of IZZI 2 are constructed by partitioning non-square rhombi into quadrants along the diagonals. The quadrants are then assigned one of four colors in such a way that no color is used twice on the same tile. Up to 180 degree rotation, there are 4*3*2/2=12 possible tiles, as shown in Figure 8a. As with the other puzzles, when two tiles are adjacent, they must have the same color on their shared border. A hexagonal solution is shown in Figure 8b.



a) Pieces for IZZI 2



b) Hexagon solution for IZZI 2

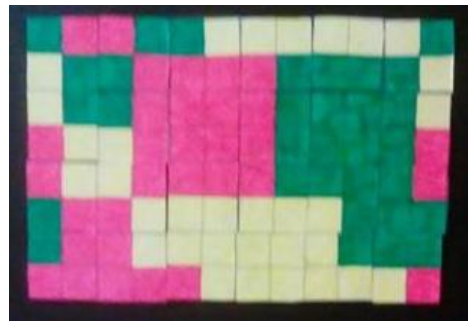
Figure 8: IZZI 2

VERTEX-MATCHING PUZZES

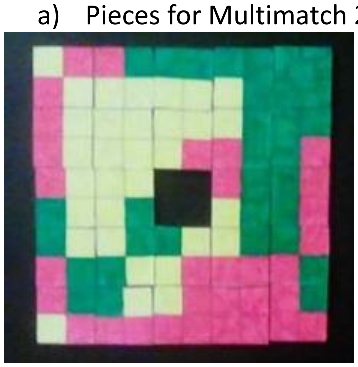
In the previous sections, the tiles were constructed by connecting each vertex to the center of the tile via the diagonal. When the sections were colored, each edge was a solid color, whereas each vertex had one or two colors. An alternate way of making tiles is to connect the midpoint of each edge to the center of the shape. Notice that this results in each vertex being a single color, whereas each edge has one or two colors. As with the first two sections, the goal is to arrange the tiles in such a way that when to tiles are adjacent, both colors along their shared edge match. Note that in these cases it is impossible to make a solid border because there are not enough pieces with a solid edge.

The first such puzzle that we will consider is *Multimatch II*. Wade Philpott conceived of the idea in 1973 and the puzzle has been produced by Kadon Enterprises since 1990. This puzzle consists of square tiles that have been divided into quadrants by connecting the midpoint of the edge to the center of the square. The set of tiles consists of every possible combination of no more than three colors, up to rotation of the tiles. As with the MacMahon Squares, this results in twenty-four distinct tiles, as shown in Figure 9a. We can consider many of the same problems as before. A 6X4 rectangular solution is given in Figure 9b. A 5X5 square with a hole is shown in Figure 9c. Note that solutions to the duplication and triplication problem can be obtained by splitting the rectangular solution in the appropriate way. With this being said, we have provided an interesting solution to the triplication problem in Figure 9d. In this solution, the colors on the three 6X2 rectangles are obtained by a cyclic permutation. Namely red to green to yellow to red.

a) Pieces for Multimatch 2



b) A 6X4 rectangular solution



c) A 5X5 square solution with a hole

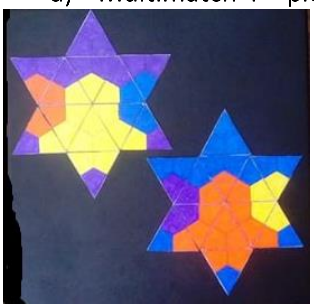
d) An interesting triplication solution

Figure 9: Multimatch II

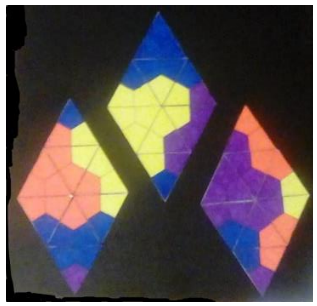
Multimatch IV is a triangular variant in which equilateral triangles have been divided into third by connecting the midpoint of each edge to the center. The sectors are then colored one of four colors in such a way that all combinations up to rotation are represented. As with MacMahon Triangles, this results in twenty-four distinct tiles, as shown in Figure 10a. As with the other variants, the goal is to make a solid shape such that if tiles are adjacent, then the colors along their shared edge match. According to the research done by Kadon Enterprises, a hexagonal solution with this set of tiles is impossible. However, a solution in which the tiles are arranged into a parallelogram is shown in Figure 10b. Solutions to the duplication and triplication problem are shown in Figure 10c and Figure 10d, respectively. Figure 10e illustrates a solution in which the tiles are arranged into four matching hexagons.

a) Multimatch 4 – pieces

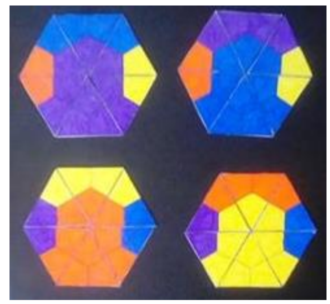
b) Parallelogram solution



c) Duplication solution – two stars



d) Triplication solution – three diamonds



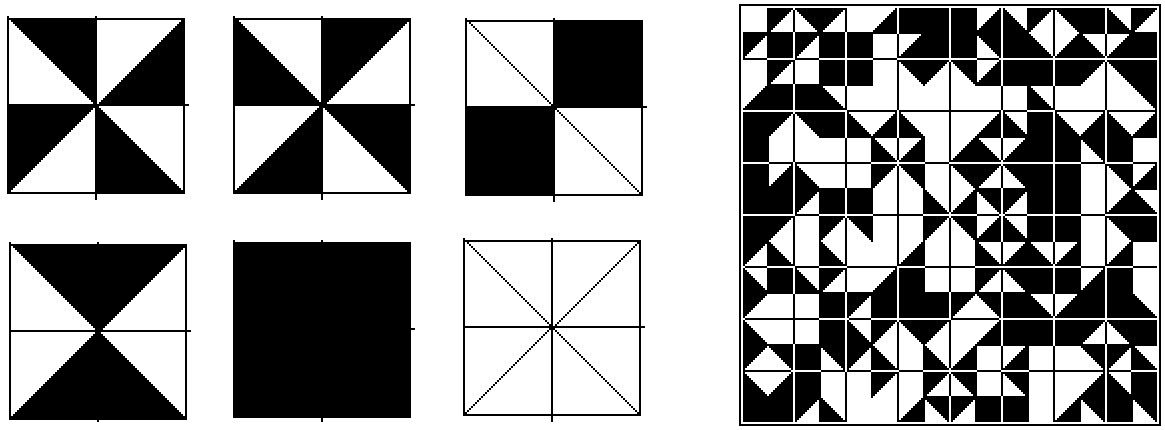
e) Four hexagons

Figure 10: Multimatch 4

HYBRID PUZZLES

The final puzzles that we consider have the potential of two colors on each edge and two colors on each vertex. For this reason, they can be thought of as “hybrids” of the two types.

The first that we consider is *IZZI*. IZZI was invented by Frank Nichols in 1992. The tiles are constructed by dividing squares into eighths by connecting opposite corners and opposite midpoints. Each of the eight sectors is assigned a color. Up to rotation, the number of distinct tiles is $\frac{1}{4}(m^8 + m^4 + 2m^2)$. When $m = 2$ this gives seventy distinct tiles. However, IZZI removes the six tiles shown in Figure 11a to bring the number of tiles to sixty-four. The goal is to arrange them into a square in such a way that if two tiles touch, the pattern along their shared edge match. One such solution is given in Figure 11b.



a) IZZI's missing tiles b) A Solution to IZZI

Figure 11: IZZI

In 2018, I created a variation using equilateral triangles. Each triangular tile is divided into sixths by drawing a line from each vertex to the opposite midpoint. Each sector is assigned one of two colors. When m colors are used, there are $\frac{1}{3}(m^6 + 2m^2)$ distinct tiles, up to rotation. When $m = 2$, this gives twenty-four distinct tiles. These tiles are shown in Figure 12a. As usual, the goal is to arrange the tiles into a solid shape such that if two tiles are adjacent, the pattern along the shared edge match. A hexagon solution is shown in Figure 12b and a parallelogram solution is given in Figure 12c.

a) Pieces for my variation b) A hexagon solution c) A parallelogram solution

Figure 12: A variation which consists of all 2-colorings of triangles partitioned into six sectors

FINAL THOUGHTS

While working with the MacMahon type puzzles, I was particularly impressed with two aspects of these puzzles. First, the puzzles can be made relatively easily using supplies found at most craft stores. Personally, I have made several sets using cardstock and glue. Most of the above pictures are of sets that I made using adhesive magnets, printed patterns, and markers. MacMahon himself suggested making sets out of painted wood. As a bonus, these make an excellent (and beautiful) weekend art project. This aspect makes them ideal as an activity in math circles or extracurricular clubs.

Second, these puzzles are remarkably robust. In the booklets produced by Kadon Enterprises, they present dozens of puzzles for each of the tile sets they produce. In fact, MacMahon's book considers additional variants as well. Some of these are simply reductions of the existing tile sets. Others are entirely new puzzles. While MacMahon only consider solitaire puzzles, Kadon gives several games for two to four players. There is no doubt that additional puzzles in the spirit of MacMahon can made and analyzed.

ACKNOWLEDGEMENTS

The author wishes to acknowledge Kate Jones of Kadon Enterprises for providing information and solutions to several of the puzzles.

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