Peg Solitaire on Graphs: Results, Variations, and Open Problems

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Acknowledgements

I like to acknowledge my peg solitaire co-authors:
Aaron Gray, Hannah Green, Russell Harper, Paul Hoilman,
Tony Rodriguez, and Clayton Walvoort.

Description of the game

Peg solitaire is a table game which traditionally begins with “pegs” in every space except for one which is left empty (in other words, a “hole”). If in some row or column two adjacent pegs are next to a hole (as in Figure 1), then the peg in x can jump over the peg in y into the hole in z. The peg in y is then removed. The goal is to remove every peg but one. If this is achieved, then the board is considered solved. For more information on traditional peg solitaire, refer to Beasley [1] or Berlekamp et al. [10]

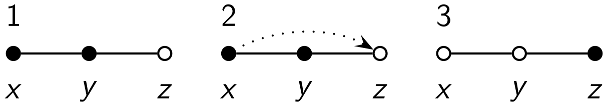


Figure: A Typical Jump in Peg Solitaire

A Brief History (part 1)

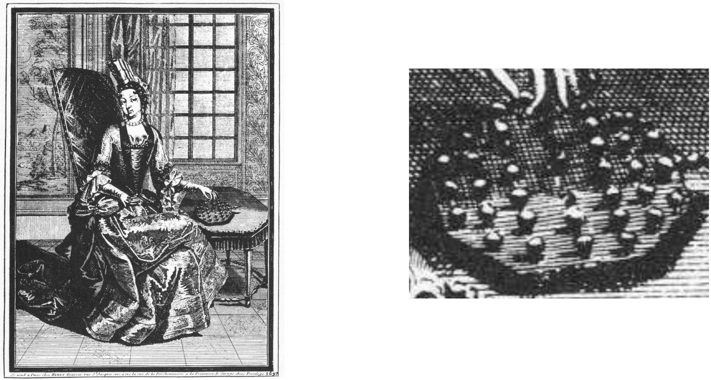


Figure: *Madame la Princesse de Soubise joüant au jeu de Solitaire* by Claude-Auguste Berey, 1697.

A Brief History (part 2)

Not so very long ago there became widespread an excellent kind of game, called *Solitaire*, where I play on my own, but as with a friend as witness and referee to see that I play correctly. A board is filled with stones set in holes, which are removed in turn, but none (except the first, which may be chosen for removal at will) can be removed unless you are able to jump another stone across it into an adjacent empty place, when it is captured as in Draughts. He who removes all the stones right to the end according to this rule, wins; but he who is compelled to leave more than one stone still on the board, yields the palm.

Gottfried Wilhelm Leibniz,
Miscellanea Berolinensia **1** (1710) 24.

The version you are most likely familiar with...

The generalization to graphs

In a 2011 paper (B-Hoilman [6]), the game is generalized to graphs in the combinatorial sense. So, if there are pegs in vertices x and y and a hole in z, then we allow x to jump over y into z provided that $xy \in E$ and $yz \in E$. The peg in y is then removed.

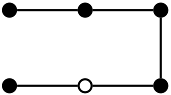
In particular, we allow 'L'-shaped jumps, which are not allowed in the traditional game.

Definitions from [6]

- A graph G is *solvable* if there exists some vertex s so that, starting with $S = \{s\}$, there exists an associated terminal state consisting of a single peg.
- A graph G is *freely solvable* if for all vertices s so that, starting with $S = \{s\}$, there exists an associated terminal state consisting of a single peg.
- A graph G is *k-solvable* if there exists some vertex s so that, starting with $S = \{s\}$, there exists an associated minimal terminal state consisting of k nonadjacent pegs.
- In particular, a graph is *distance 2-solvable* if there exists some vertex s so that, starting with $S = \{s\}$, there exists an associated terminal state consisting of two pegs that are distance 2 apart.

Examples

If n is even, then the path P_n is solvable, but not freely solvable.



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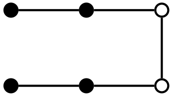

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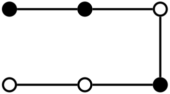

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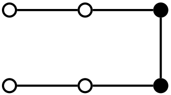

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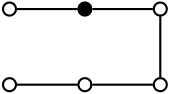

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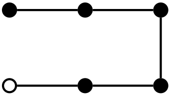

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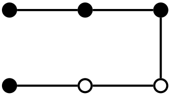

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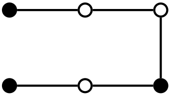
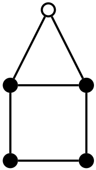


Examples (Part 2)

The “house” graph is freely solvable.



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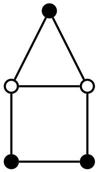

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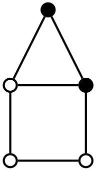

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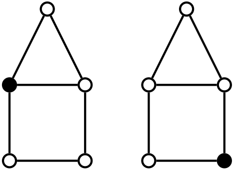
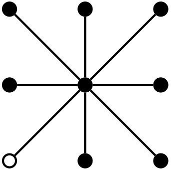


It may seem like I cheated, but I didn't!

Examples (Part 3)

For $k \geq 2$, the star, $K_{1,k+1}$ is k-solvable.



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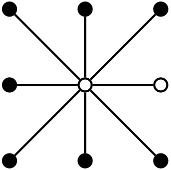
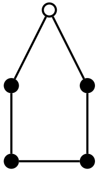


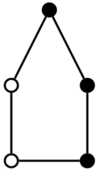
Examples (Part 4)

If n is odd and $n \geq 5$, then the cycle C_n is distance 2-solvable.



Examples (Part 4)

If n is odd and $n \geq 5$, then the cycle C_n is distance 2-solvable.



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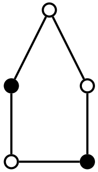


Tricks of the Trade

The Inheritance Principle -
If G is a k-solvable spanning subgraph of H, then H is (at worst) k-solvable.

Tricks of the Trade (Part 2)

The Duality Principle -
Let T be a terminal configuration of pegs associated with starting configuration S. If S' and T' are obtained from S and T, respectively, by reversing the roles of pegs and holes, then S' is a terminal state associated with starting state T'.

Common Graphs [6]

The usual goal is to determine the necessary and sufficient conditions for the solvability of a family of graphs. To date, the solvability of the following graphs has been determined:

- $K_{1,n}$ is $(n-1)$-solvable; $K_{n,m}$ is freely solvable for $n, m \geq 2$.
- P_n is freely solvable iff $n = 2$; P_n is solvable iff n is even or $n = 3$; P_n is distance 2-solvable in all other cases.
- C_n is freely solvable iff n is even or $n = 3$; C_n is distance 2-solvable in all other cases.
- The Petersen Graph, the platonic solids, the archimedean solids, the complete graph, and the n-dimensional hypercube are freely solvable.

Common Graphs (Part 2)

- The double star $DS(L, R)$ is freely solvable iff $L = R$ and $R \neq 1$; $D(L, R)$ is solvable iff $L \leq R + 1$; $DS(L, R)$ is distance 2-solvable iff $L = R + 2$; $DS(L, R)$ is $(L - R)$-solvable in all other cases [7].
- The solvability of all graphs with seven vertices or less [3].

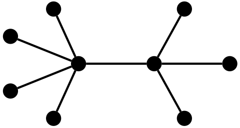


Figure: The Double Star - $DS(4, 3)$

Cartesian Products

One of the most important results from B-Hoilman [6] was the following:

Theorem

(i) If G and H are both solvable graphs, then the Cartesian product $G \square H$ is solvable.

(ii) If G is solvable and H is distance 2-solvable, then $G \square H$ is solvable.

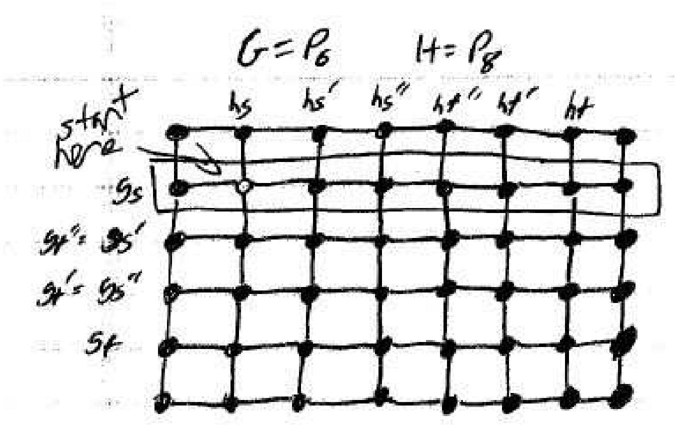
(iii) If G and H are both distance 2-solvable, then $G \square H$ is solvable.

Cartesian Products (Part 2)

The following observation is useful for solving Cartesian products:

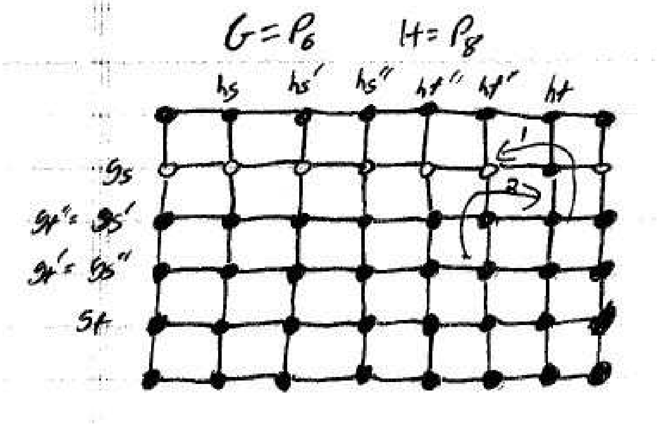
Suppose that G has at least three vertices and is k-solvable beginning with initial hole in g_s. Assuming that a jump is possible, then there is a first jump, say from g_s'' over g_s' into g_s. It follows that if G has holes in g_s' and g_s'' and pegs everywhere else, then G is k-solvable from this state. Similarly, if G is solvable with the final peg in g_t, then there is a final jump, say from g_t'' over g_t' into g_t.

Cartesian Products - The Proof



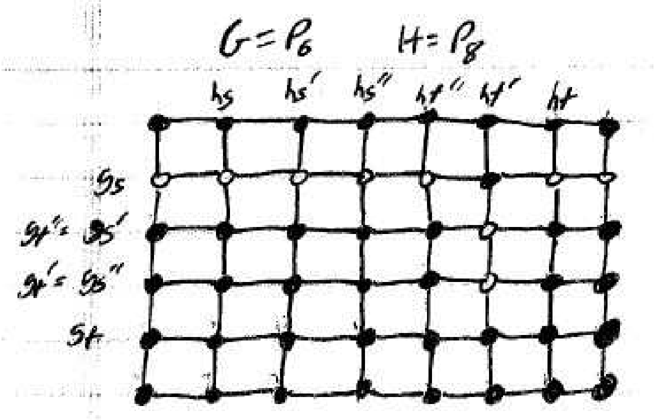
Solve one copy of H.

Cartesian Products - The Proof (Part 2)



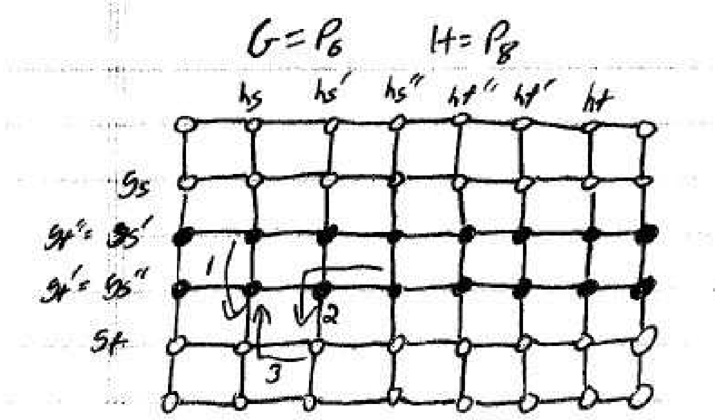
Do some local corrections.

Cartesian Products - The Proof (Part 3)



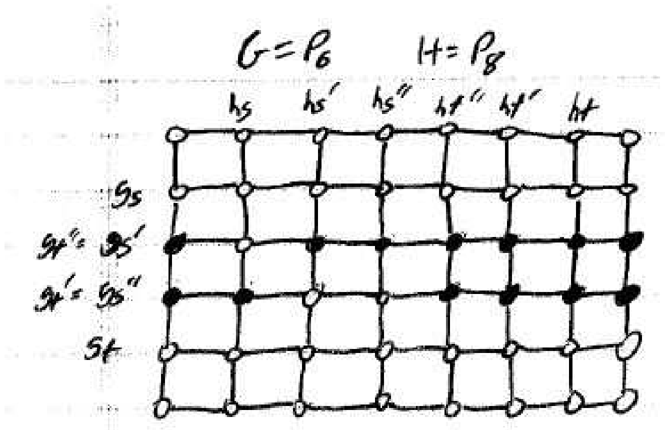
Notice that on each copy of G, we either have a hole in the right place for a solution, or two holes after the first jump in a solution. "Solve" each copy, but stop short of making the final jump.

Cartesian Products - The Proof (Part 4)



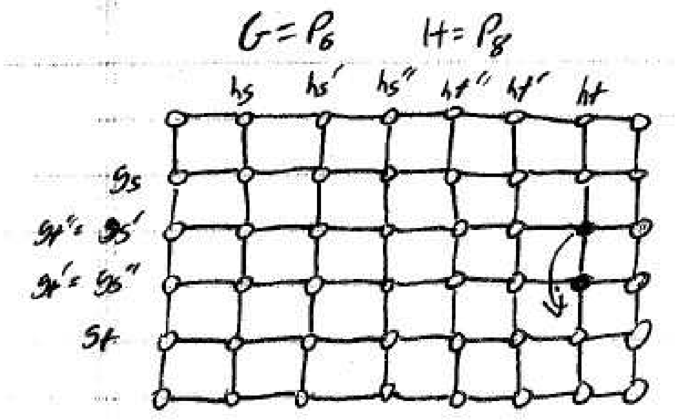
Make some more local corrections.

Cartesian Products - The Proof (Part 5)



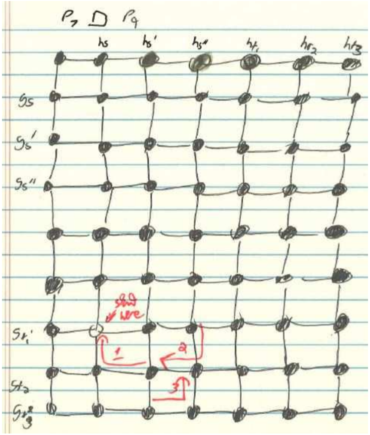
Each copy of H has a hole in the in the right place for a solution, or two holes where the first jump in a solution would be. So solve them.

Cartesian Products - The Proof (Part 6)



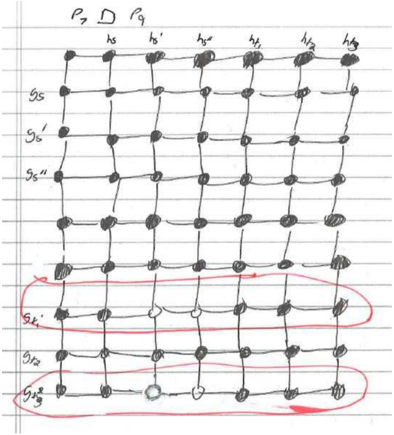
One final jump to solve the graph.

Cartesian Products - Distance 2-solvable



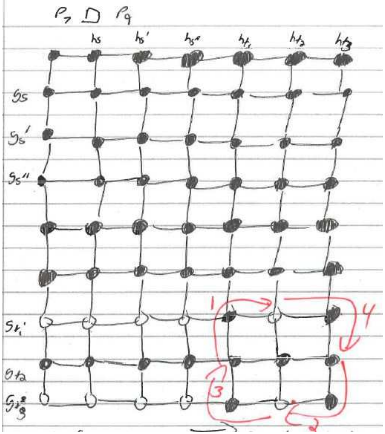
Notice that here we start with a hole where one of the final two pegs would be in a distance 2-solution on G. Then do some local corrections.

Cartesian Products - Distance 2-solvable (Part 2)



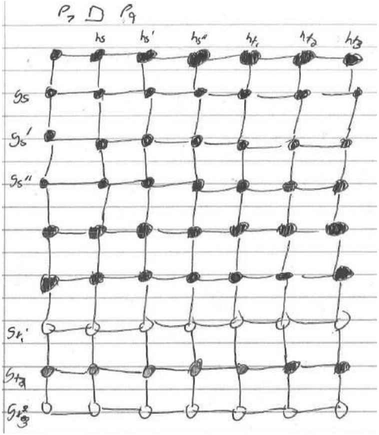
These two copies of H have holes in the correct places (after first jump). Distance 2-solve them independently.

Cartesian Products - Distance 2-solvable (Part 3)



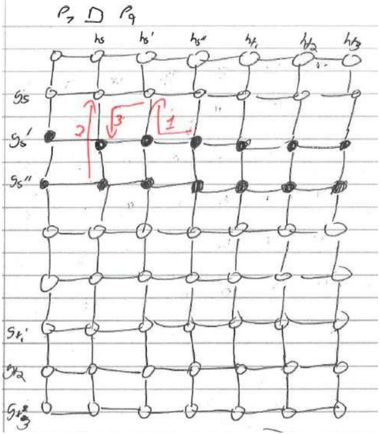
Do another round of local corrections.

Cartesian Products - Distance 2-solvable (Part 4)



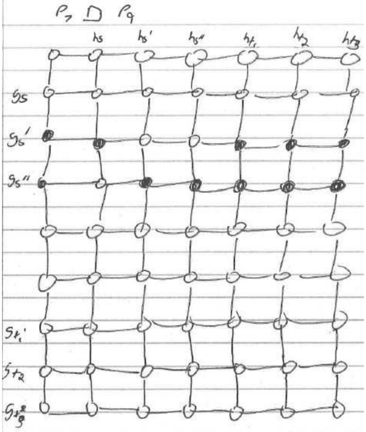
Now, each copy of G has holes in the terminal positions for a distance 2-solution. Thus, we can use the Duality Principle to solve each copy of G. Again, on each copy, stop short of the final jump.

Cartesian Products - Distance 2-solvable (Part 5)



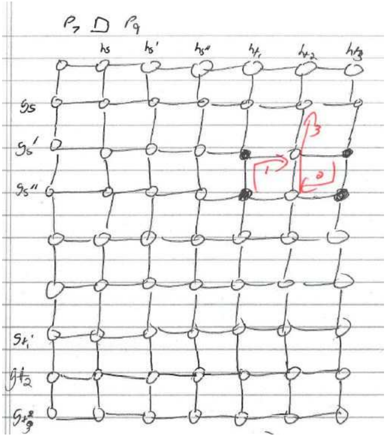
Do more local corrections.

Cartesian Products - Distance 2-solvable (Part 6)



Each copy of H has holes in the right place. Distance 2-solve them independently.

Cartesian Products - Distance 2-solvable (Part 7)



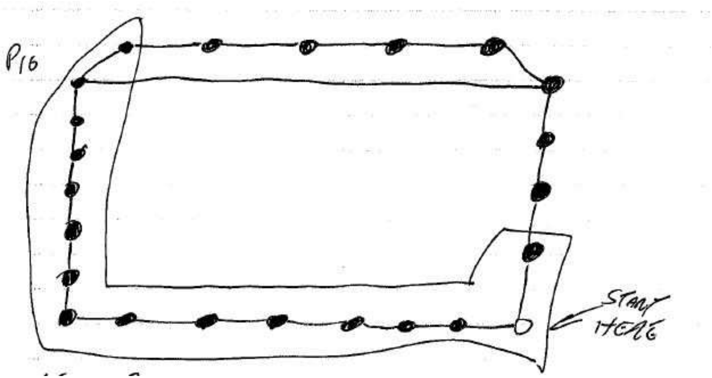
Solve the graph with a final round of jumps.

Chorded Odd Cycle

Consider the cycle on n vertices, where the vertices are labeled with the elements of $\mathbb{Z}_n$ in the obvious way. Recall that even cycles are freely solvable and that odd cycles are distance 2-solvable. What if we add an edge between vertices 0 and m? This graph is denoted $C(n, m)$.

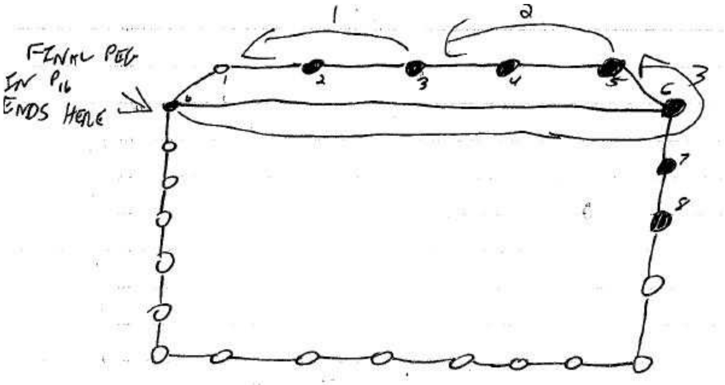
Theorem [3] For all n and $m \leq n$, the chorded odd cycle $C(2n + 1, m)$ is solvable.

Chorded Odd Cycle - The Proof



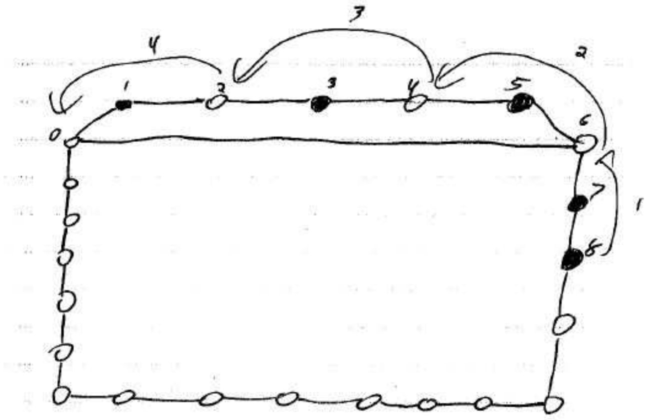
Because the cycle has odd length, on one side of the chord we have an even path. The final peg on an even path will be in the next to last vertex.

Chorded Odd Cycle - The Proof (Part 2)



Make a series of jumps on the other side of the chord.

Chorded Odd Cycle - The Proof (Part 3)



Because we saved a couple of pegs, we can now “hopscotch” to remove the remaining pegs.

Other Results About Chorded Cycles

Some other results about cycles:

- For all n, $C(n, 2)$ is freely solvable [2].
- If $n \leq 9$ and $m \leq n$, then $C(2n + 1, m)$ is freely solvable [3].

Conjecture - All chorded odd cycles freely solvable.

Edge Critical Graphs

Notice that the addition of *any* edge to the odd cycle changes its solvability. Hence, we say that it is an *edge critical* graph. These graphs were studied by B-Gray [4]. Some natural questions include:

(i) What other graphs are edge critical graphs?

(ii) How much can edge addition improve the solvability of a graph?

Extremal results

Determining edge critical graphs is related to the extremal problem. To motivate this, note the following:

It seems to be the case that most graphs are freely solvable. In fact, of the 996 connected non-isomorphic graphs on seven vertices or less, only 54 are not freely solvable [3]. It seems counterintuitive, but perhaps the *unsolvable* graphs are more interesting.

Naturally, we expect that as the number of edges in a connected graph increase, the more likely it is to be solvable or freely solvable.

Define $\tau(n)$ to be the maximum number of edges in an unsolvable connected graph on n vertices.

Extremal results (part 2)

It order to attack this problem, B-Gray [4] introduced the *hairy complete graph*. The hairy complete graph is obtained from the complete graph K_n by appending a_i pendants to the ith vertex of the complete graph. Without loss of generality, $a_1 \geq ... \geq a_n$ and $a_1 \geq 1$. This graph is denoted $K_n(a_1, ..., a_n)$.

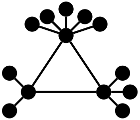


Figure: The hairy complete graph $K_3(5, 3, 2)$.

Extremal results (part 2)

Theorem [4]

For the hairy complete graph $G = K_n(a_1, ..., a_n)$:

(i) The graph G is solvable iff $a_1 \leq \sum_{i=2}^{n} a_i + n - 1$;

(ii) The graph G is freely solvable iff $a_1 \leq \sum_{i=2}^{n} a_i + n - 2$ and $(n, a_1, a_2, a_3) \neq (3, 1, 0, 0)$;

(iii) The graph G is distance 2-solvable iff $a_1 = \sum_{i=2}^{n} a_i + n$;

(iv) The graph G is $(a_1 - \sum_{i=2}^{n} a_i - n + 2)$-solvable if $a_1 \geq \sum_{i=2}^{n} a_i + n$.

Extremal results (part 3)

In particular, $K_n(n, 0, ...0)$ and $K_n(n+1, 0, ..., 0)$ are not solvable. However, the addition of *any* edge to either of these graphs results in a solvable graph. Hence, they are edge critical. Note that the number of edges in $K_n(n, 0, ...0)$ is $n(n+1)/2$. From this it follows that if n is even, then $\tau(n) \geq n(n+2)/8$.

Conjecture - If n is even, then $\tau(n) = n(n+2)/8$ and $K_{n/2}(n/2, 0, ..., 0)$ is the extremal graph.

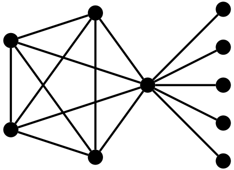


Figure: $K_5(5, 0, 0, 0, 0)$

Classification of trees

If G is a (freely) solvable subgraph of H, then H is (freely) solvable. Since every connected graph has a spanning subtree, the Inheritance Principle implies that a natural (and very important) problem is to determine which trees are solvable. All trees of diameter three or less were classified in [6, 7]. B-Walvoort took the next natural step by classifying the solvability of trees of diameter four [9].

Parametrization

Diameter 4 trees will be parameterized as $K_{1,n}(c; a_1, \ldots, a_n)$, where n is the number of non-central support vertices, c is the number of pendants adjacent to x and a_i is the number of pendants adjacent to y_i. Without loss of generality, we may assume that $a_1 \geq ... \geq a_n \geq 1$.

Also, let $k = c - s + n$, where $s = \displaystyle\sum_{i=1}^{n} a_i$.

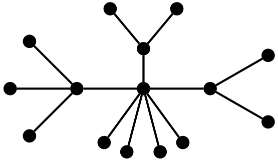


Figure: The graph $K_{1,3}(4; 3, 2, 2)$

Trees of diameter 4

Theorem 1 [9]

Assume $a_1 \geq 2$. The conditions for solvability of such diameter four trees are as follows:

(i) The graph $K_{1,n}(c; a_1, \ldots, a_n)$ is solvable iff $0 \leq k \leq n+1$.

(ii) The graph $K_{1,n}(c; a_1, \ldots, a_n)$ is freely solvable iff $1 \leq k \leq n$.

(iii) The graph $K_{1,n}(c; a_1, \ldots, a_n)$ is distance 2-solvable iff $k \in \{-1, n+2\}$.

(iv) The graph $K_{1,n}(c; a_1, \ldots, a_n)$ is $(1-k)$-solvable if $k \leq -1$. The graph $K_{1,n}(c; a_1, \ldots, a_n)$ is $(k-n)$-solvable if $k \geq n+2$.

More on trees of diameter 4

Theorem 2 [9]

The conditions for solvability of $K_{1,n}(c; 1, ..., 1)$ is as follows:

(i) The graph $K_{1,2r}(c; 1, \ldots, 1)$ is solvable iff $0 \leq c \leq 2r$ and $(r, c) \neq (1, 0)$. The graph $K_{1,2r+1}(c; 1, \ldots, 1)$ is solvable iff $0 \leq c \leq 2r + 2$.

(ii) The graph $K_{1,n}(c; 1, \ldots, 1)$ is freely solvable iff $1 \leq c \leq n - 1$.

(iii) The graph $K_{1,2r}(c; 1, \ldots, 1)$ is distance 2-solvable iff $c = 2r + 1$ or $(r, c) = (1, 0)$. The graph $K_{1,2r+1}(c; 1, \ldots, 1)$ is distance 2-solvable iff $c = 2r + 3$.

(iv) The graph $K_{1,2r}(c; 1, \ldots, 1)$ is $(c - 2r + 1)$-solvable if $c \geq 2r + 1$. The graph $K_{1,2r+1}(c; 1, \ldots, 1)$ is $(c - 2r - 1)$-solvable if $c \geq 2r + 3$.

The solvability of caterpillars

The caterpillar can be obtained from the path on n vertices by appending a_i pendants to ith vertex on the path. Such a caterpillar is denoted $P_n(a_1, ..., a_n)$. Note that we can assume that $a_1 \geq a_n$. If $a_1 = a_n$, then we can assume $a_2 \geq a_{n-1}$ and so on to ensure a unique parameterizations under this notation.

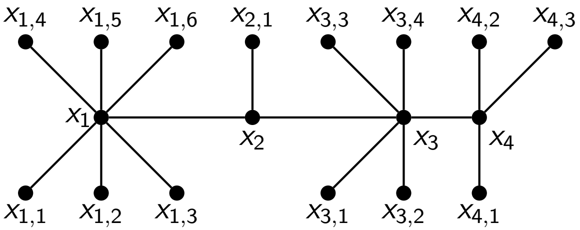


Figure: The caterpillar $P_4(6, 1, 4, 3)$

The solvability of caterpillars (part 2)

B, Green, and Harper determined the solvability of several infinite classes of caterpillars. For example, they determine the solvability of caterpillars of the form $P_4(a_1, a_2, a_3, a_4)$ is as follows:

Theorem [5]

(I) The caterpillar $P_4(a_1, a_2, a_3, a_4)$ with $a_1 \geq a_2 + 1$ is solvable if and only if one of the following is true: *(i)* $a_1 = a_2 + 1$ and either $a_3 - a_4 \leq 1$ or $a_4 - a_3 \leq 3$; *(ii)* $a_1 = a_2 + 2$ and $a_4 - a_3 \leq 2$; *(iii)* $a_1 = a_2 + 2$, $a_2 \geq 1$, and $a_3 - a_4 \geq 0$; *(iv)* $a_1 = a_2 + 3$ and $a_3 = a_4 - 1$.

(II) The caterpillar $P_4(a_1, a_2, a_3, a_4)$ with $a_2 = a_1 + m$, where $m \geq 0$ is solvable if and only if one of the following is true:
(i) $a_3 = a_4 + k$, where $k \geq 0$ and $-2 \leq m - k \leq 2$;
(ii) $a_4 = a_3 + k$, where $k \geq 1$ and $m + k \leq 2$.

The number of solvable trees

An interesting question involves the percentage of trees that are solvable. Suppose that T_n is the number of non-isomorphic trees on n vertices and that S_n is the number of solvable non-isomorphic trees on n vertices.

Conjecture - $S_n/T_n \geq .5$ for $n \geq 9$.

Stronger Conjecture - S_n/T_n is an "increasing" sequence for "large" n.

n	6	7	8	9	10	11	12
S_n	3	6	11	24	58	122	315
T_n	6	11	23	47	106	235	531
S_n/T_n	.5	.55	.48	.51	.55	.52	.59

Variations

There are a number of natural variations for peg solitaire on graphs. These include:

(i) Fool's Solitaire

(ii) Peg Duotaire

(iii) Reversible Peg Solitaire

(iv) Merging Peg Solitaire

(v) Bridge Burning Solitaire

Fool's Solitaire

In *fool's solitaire*, the player tries to leave the maximum number of pegs possible under the caveat that the player jumps whenever possible. This maximum number will be denoted $Fs(G)$.

If G is a connected graph, then a sharp upper bound for the fool's solitaire number is $Fs(G) \leq \alpha(G)$, where $\alpha(G)$ denotes the independence number of G [8].

Known Results from [8]

B-Rodriguez determined the fool's solitaire number for several families of graphs. In particular:

- $Fs(K_{1,n}) = n$.
- $Fs(K_{n,m}) = n - 1$ if $n \geq m \geq 2$.
- $Fs(P_n) = \lfloor n/2 \rfloor$.
- $Fs(C_n) = \lfloor \frac{n-1}{2} \rfloor$.
- $Fs(Q_n) = 2^{n-1} - 1$.
- The fool's solitaire number for all connected graphs with six vertices or less.

A Natural Conjecture

In all of the above cases, $Fs(G) \geq \alpha(G) - 1$. In fact, of the 143 non-isomorphic connected graphs with six vertices or less, 130 satisfy $Fs(G) = \alpha(G)$. So a natural conjecture is that

$$\alpha(G) - 1 \leq Fs(G) \leq \alpha(G).$$

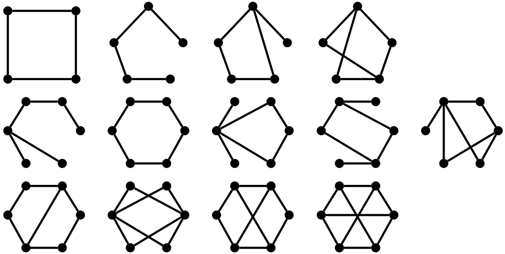


Figure: Graphs with $n(G) \leq 6$ such that $Fs(G) = \alpha(G) - 1$

A Counterexample...

However, trees of diameter four provide an infinite class of counterexamples to the above conjecture.

Theorem 3 [9]

Consider the diameter four tree $G = K_{1,n}(c; a_1, \ldots, a_n)$, where $a_i \geq 2$ for $1 \leq i \leq n - \ell$, $a_i = 1$ for $n - \ell + 1 \leq i \leq n$, and $n \geq 2$.

(i) If $c = 0$ and $\ell = 0$, then $Fs(G) = s + c - \left\lfloor \frac{n}{3} \right\rfloor$.

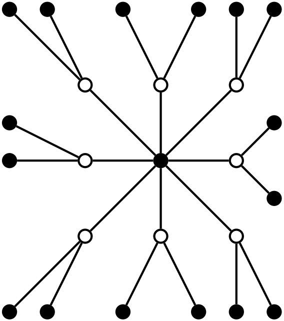
(ii) If $c \geq 1$ and $\ell = 0$, then $Fs(G) = s + c - \left\lfloor \frac{n+1}{3} \right\rfloor$.

(iii) If $\ell \geq 1$, then $Fs(G) = s + c - \left\lfloor \frac{n-2m+1}{3} \right\rfloor$, where $m = \min\{\ell, \left\lfloor \frac{n}{2} \right\rfloor\}$.

Note that the difference between $\alpha(G)$ and $Fs(G)$ can be arbitrarily large in trees of diameter four. However, $Fs(G) > 5\alpha(G)/6$ for all such trees.

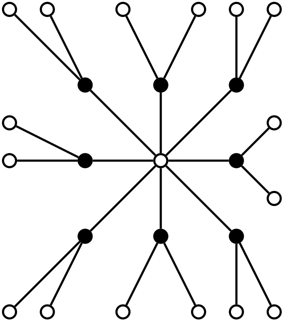
A Counterexample... (Proof)

Consider the maximum independent set for a tree of diameter four.



A Counterexample... (Proof)

To show that this is not a fool's solitaire solution, consider the dual configuration.



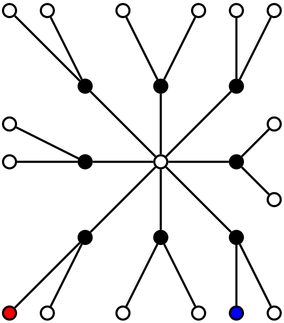
A Counterexample... (Proof)

The dual configuration had no adjacent pegs. So it isn't solvable. By the Duality Principle, the maximum independent set is *not* achievable as the fool's solitaire solution.

Thus, we must add pegs to the dual. By the Duality Principle, this is equivalent to removing pegs from the independent set.

A Counterexample... (Proof)

Here we've added two pegs to the dual. These are colored red and blue.



A Counterexample... (Proof)

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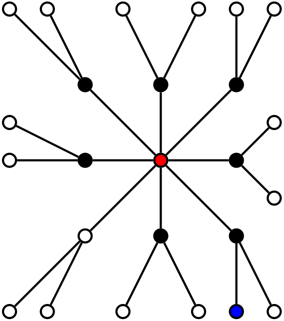

A Counterexample... (Proof)

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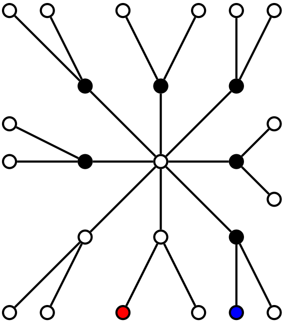

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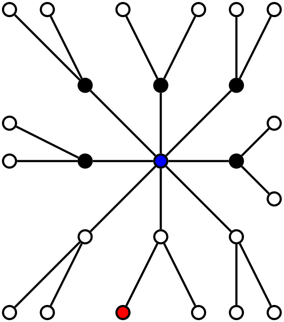

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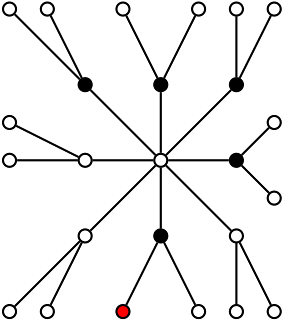

A Counterexample... (Proof)

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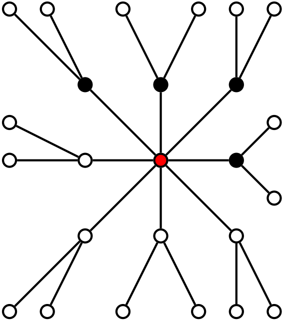
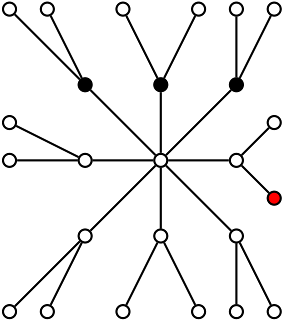


A Counterexample... (Proof)

Here we've added two pegs to the dual. These are colored red and blue.



Notice that we could remove two pegs with the first added peg and three with the second. This is why we end up with the floor of $n/3$.

More Fool's Solitaire Results

Loeb and Wise [15] extended fool's solitaire results to joins and Cartesian products. In particular, they showed:

(i) $Fs(G + K_1) = \alpha(G + K_1)$.

(ii) If $|V(G)|, |V(H)| \geq 2$ and $|E(G)| + |E(H)| \geq 1$, then $Fs(G + H) = \alpha(G + H)$.

(iii) If $n \geq 3$, then $Fs(G \square K_n) = \alpha(G \square K_n)$.

(iv) If G and H are solvable with the initial hole in any v such that the final peg is in the closed neighborhood of v, then $Fs(G \square H) \geq Fs(G)Fs(H)$.

Open Problems for Fool's Solitaire

- What other graphs have $Fs(G) < \alpha(G) - 1$?
- Is there a non-trivial lower bound on $Fs(G)$?
- For what graphs does edge deletion lower the fool's solitaire number?
- How much can edge deletion lower the fool's solitaire number?

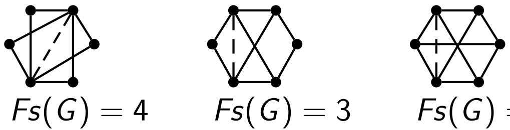


Figure: Graphs in which Edge Deletion Lowers $Fs(G)$

Duotaire

Peg duotaire is played between two players. The first player selects the initial hole. The players then alternate making peg solitaire moves on the board. The last player to make a jump wins. For information on traditional peg duotaire see [14, 16].
B-Gray are currently investigating the peg duotaire on graphs.

Duotaire - A Competitive Parameter

As a variation, suppose duotaire is played between the *maximizer* and the *minimizer*. The maximizer (minimizer) strives to make the cardinality of the terminal set as large (small) as possible. When both players make optimal choices, the cardinality of the resulting terminal set is fixed. This results in a *competitive graph parameter* (see Phillips and Slater [17, 18]).

When the maximizer (minimizer) plays first, we denote this parameter $D^+(G)$ ($D^-(G)$).

An Observation About the Duotaire Parameters

Let $Ps(G)$ denote the minimum number of pegs that can be left on the graph G. Recall that $Fs(G)$ denotes the maximum number of pegs that can be left on the graph G. The following inequalities are immediate:

$$Ps(G) \leq D^-(G) \leq Fs(G)$$
$$Ps(G) \leq D^+(G) \leq Fs(G).$$

Interesting Results About the Duotaire Parameters (Part 1)

Theorem - For the path on n vertices, P_n,

(i) $D^{-}(P_n) = Ps(P_n) \in \{1, 2\}$

(ii) $D^{+}(P_n) = Fs(P_n) = \lfloor n/2 \rfloor$.

So, the difference $D^{+}(G) - D^{-}(G)$ can be made arbitrarily large!

Interesting Results About the Duotaire Parameters (Part 2)

Theorem - For the complete bipartite graph $K_{n,m}$ with partitions of size n and m, $D^{-}(K_{n,m}) = n - m$ and $D^{+}(K_{n,m}) = n - m + 1$.

Note that $n \geq m \geq 2$, $Ps(K_{n,m}) = 1$ and $Fs(K_{n,m}) = n - 1$ [6, 8]. Hence, for any $k, \ell \in \mathbb{Z}^{+}$, there exists a graph such that $D^{-}(G) - Ps(G) = k$ and $Fs(G) - D^{+}(G) = \ell$.

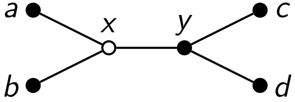
Interesting Results About the Duotaire Parameters (Part 3)

Theorem - For the double star $DS(n, m)$, if $n \geq m \geq 2$, then $D^{-}(DS(n, m)) = n + m - 2$ and $D^{+}(DS(n, n)) = n + m - 3$.

Note that this means that both players actually do better when they play second!

Why?

Suppose the minimizer goes first and selects the leaf a as the initial hole. Then the maximizer can jump y over x into a. So instead, the minimizer should choose x as their initial hole.



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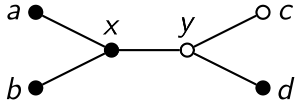
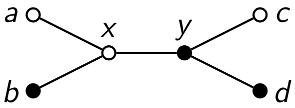


Maximizer's first jump is forced.

Why?

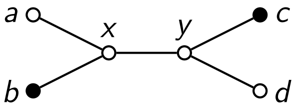
Suppose the minimizer goes first and selects the leaf a as the initial hole. Then the maximizer can jump y over x into a. So instead, the minimizer should choose x as their initial hole.



After the minimizer's first jump.

Why?

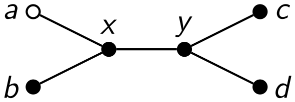
Suppose the minimizer goes first and selects the leaf a as the initial hole. Then the maximizer can jump y over x into a. So instead, the minimizer should choose x as their initial hole.



The maximizer ends the game.

Why?

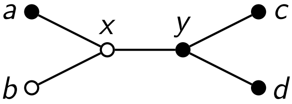
Suppose the maximizer goes first and selects the leaf a as the initial hole.



The minimizer clearly doesn't want to jump y over x into a.

Why?

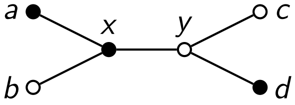
Suppose the maximizer goes first and selects the leaf a as the initial hole.



After the minimizer's first jump.

Why?

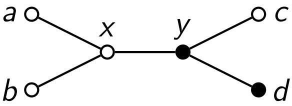
Suppose the maximizer goes first and selects the leaf a as the initial hole.



The maximizer's first jump is forced.

Why?

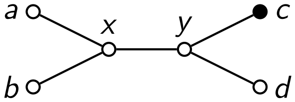
Suppose the maximizer goes first and selects the leaf a as the initial hole.



After the minimizer's second jump.

Why?

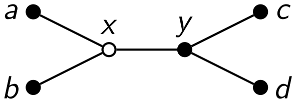
Suppose the maximizer goes first and selects the leaf a as the initial hole.



The maximizer ends the game.

Why?

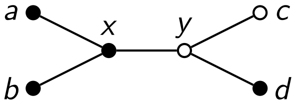
Suppose the maximizer goes first and instead selects x as the initial hole.



The minimizer's first move is forced.

Why?

Suppose the maximizer goes first and instead selects x as the initial hole.



After the minimizer's first move, this reduces to the earlier case.

Open Problems About the Duotaire Parameter

(i) Characterize those graphs where $D^{-}(G) > D^{+}(G)$.

(ii) For all $k \in \mathbb{Z}^{+}$, find a graph G such that $D^{-}(G) - D^{+}(G) = k$.

(iii) Characterize those graphs where $D^{-}(G) = D^{+}(G)$.

(iv) Characterize those graphs where $D^{-}(G) = Ps(G)$.

(v) Characterize those graphs where $D^{+}(G) = Fs(G)$.

Another variation

Engbers and Stocker [12] consider a variation of peg solitaire on graphs in which we allow *unjumps* in addition to our regular jumps. Namely, if there is a peg in x, holes in y and z, and $xy, yz \in E(G)$, then we can jump from x over y into z. This restores the peg in y.

Naturally, we want to know which graphs are solvable in this new variation.

Another variation (part 2)

Engbers and Stocker [12] complete characterize the graphs that are solvable in their variation. Namely:

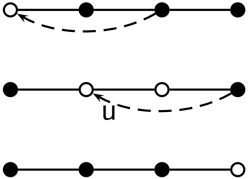
(i) The star $K_{1,n}$ is still unsolvable for $n \geq 3$.

(ii) If G is a non-star graph with maximum degree at least 3, then G is freely solvable.

(iii) P_n and C_n are solvable iff n is divisible by 2 or 3.

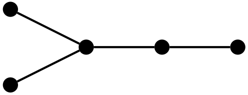
Reversible Solitaire - Freely Solvable

Showing that non-star graphs with maximum degree at least 3 is solvable involves two steps.
First, show that we can do a P_4-move:



Reversible Solitaire - Freely Solvable (Part 2)

Use the P_4-moves to move pegs onto the “widget” below. Then remove them on the widget!



Reversible Solitaire - Paths

To show that paths are unsolvable (except when n is divisible by 2 or 3), "weight" any vertex with a 1 if it has a hole. If the vertex has a peg, then use quaternions:

(i) Weight the vertex v_ℓ i if $\ell \equiv 0 \pmod 3$.

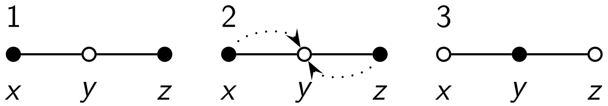
(ii) Weight the vertex v_ℓ j if $\ell \equiv 1 \pmod 3$.

(iii) Weight the vertex v_ℓ k if $\ell \equiv 2 \pmod 3$.

The weight of the configuration is the product of the weights of the vertices in left to right order. Observe that neither a jump nor an unjump will change the weight of the configuration. The configuration is solvable iff its total weight is i, j, or k. Then look at cases modulo 6.

Variation - Merging

Engbers and Weber [13] consider a variation in which the move is replaced by a "merge." If there are pegs in x and z, a hole in y, and $xy, yz \in E$, then we can *merge* the pegs in x and z onto the hole in y.



Merging - Results

Again, the goal is to determine which graphs are solvable in this variation. Some of the results from Engbers and Weber [13].

(i) The star $K_{1,n}$ is still not solvable.

(ii) If $n \geq 2$, then the path P_n is solvable.

(iii) The double star $DS(n, m)$ is solvable iff $|n - m| \leq 1$.

Yet another variation

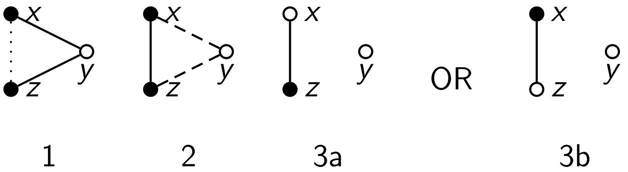
Bullington [11] considers a variation in which a move is defined as:

(i) Suppose that 'peg' vertices x and z are adjacent to a 'hole' vertex y.

(ii) Add an edge between x and z if there is not one there already.

(iii) Delete the edges xy and yz.

(iv) Choose either x or z to be a 'hole' vertex.



Yet another variation (part 2)

Bullington [11] proves the following:

(i) All paths are solvable.

(ii) All traceable and hypotraceable graphs are solvable.

(iii) $K_{n,m}$ is solvable for $n, m \geq 2$.

(iv) If v is a vertex with $deg(v) = m$ such that $G - v$ has at least m connected components, then G is not solvable.

Final Thoughts

Peg solitaire on graphs is an area with many open (and difficult) problems.

There are a number of variations that one can consider.

Questions?

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