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Statement of Research

My research has focused on graph theory and combinatorics. Currently, my research has yielded forty-seven publications, including a textbook *How to Count* published by Springer and three encyclopedia articles published in the *Encyclopedia of Mathematics and Society*. My work has appeared in peer reviewed journals such as *Congressus Numerantium* (ten papers), *The Bulletin of the Institute of Combinatorics and its Applications* (seven papers), *The Australasian Journal of Combinatorics* (four papers), *The Journal of Combinatorial Mathematics and Combinatorial Computing* (two papers), *Involve* (two papers), *The Mathematical Scientist* (one paper), *Mathematics Magazine* (one paper), *Graphs and Combinatorics* (one paper), *Discrete Applied Mathematics* (two papers), *Discussiones Mathematicae Graph Theory* (one paper), *The Art of Discrete and Applied Mathematics* (one paper), and *Discrete Mathematics* (one paper). In particular, *Graphs and Combinatorics*, *Discrete Mathematics*, *Discrete Applied Mathematics*, and *Discussiones Mathematics Graph Theory* are among the most prestigious and selective journals in the field. I am honored to have published twenty-one of these papers with ETSU student co-authors. In addition, I have also published seven articles co-authored with my colleagues in the ETSU Department of Mathematics. Additionally, I have three manuscripts under review and other manuscripts in various stages of preparation.

While at ETSU, I have presented talks at Clemson University (two invited talks), Western Carolina University (two invited talks), and Lincoln Memorial University (one talk). I have been an active participant in our departmental seminars (four talks) and the informal Abstract Algebra club (two talks). In addition, I have given twelve talks at professional meetings, including the AMS-MAA joint meeting (one talk), Mathfest in Pittsburgh (one invited talk), the Southeastern International Conference on Combinatorics, Graph Theory, and Computing in Boca Raton (five talks), the AMS Special Session on Graph Theory in Huntsville, Alabama (one invited talk), the MAA special session "Engaging Activities in Coding Theory, Cryptography, and Number Theory" (one invited talk), the International Conference on Graph Theory and Applications at MTSU (one talk), the conference on Recent Advances in Graph Theory and Linear Algebra at the University of Tennessee, Chattanooga (one talk), and the Cumberland Conference on Combinatorics, Graph Theory, and Computing (one talk). I was particularly honored to be invited to be one of the keynote speakers at the Third Annual Kennesaw Mountain Undergraduate Research Conference.

At this stage in my career, the publication that I am most proud of is my textbook *How to Count*. The text was published by Springer, one of the top publishers of science and mathematical texts. This textbook is a 361 page introduction to combinatorics, a branch of mathematics that determines the number of ways that something can happen. Various examples of the problems done in my book include the number of ways of seating guests around a table according to some restriction; handing out gumballs to children who have requirements as to how many gumballs that they receive; finding the number of distinct necklaces that can be made, up to rotation and reflection; determining the number of different games of chess and Tic-Tac-Toe that can be played. As MAA reviewer Mark Hunacek notes "I don't believe I've ever seen [the enumeration of Tic-Tac-Toe] in any other textbook." Other reviews include Sullivan's review in *Choice*, where he says that my book "is an excellent introductory text on combinatorics. The author gives the right balance of theory, computation, and applications, and he presents introductory-level topics, such as the multiplication principle, binomial theorem, and distribution problems in a clear manner... Summing Up: Highly recommended." In *zbMATH*, Astrid Reifegerste notes "the author uses a clear language and often provides an easy intuitive access to abstract subjects. The presentation is well motivated, the explanations are transparent and illustrated by carefully selected examples. Each section ends with a list of well formulated exercises which make the book ideally suited for self-instruction." In addition to my own classes, the text has been used in Dr. John Little's class at College of the Holy Cross and Dr. Stefanos Aretakis's class at the University of Toronto, Scarborough. Dr. Roy Streit used the book in order to teach himself combinatorics. At various points in his study, he contacted me directly to discuss a particular research problem he was working on. The book and my personal communications to him were cited in his manuscript "Analytic combinatorics and labeling in high level fusion and multihypothesis tracking."

My research has attracted two internal grants from the ETSU Research and Development Committee. These grants awarded $8333.33 and $9970 to fund my summer research program on two occasions. I was also awarded $870 by the ETSU Instructional Development Committee for my project "Using Puzzles to Teach Concepts in Group Theory". I was the Co-PI on two NSF grants. The first was a Noyce grant (with Dr. Jeff Knisley, Dr. Daryl Stephens, and Dr. Aimme Govett) that awarded nearly $900000 to fund scholarships for potential teachers. The second, was a \$21600 grant (with Dr. Anant Godbole) for ETSU to host the 2014 Permutation Patterns conference. In 2009 and 2016, I received the Departmental Award for Outstanding Faculty in Research. I was also awarded the College of Arts and Sciences award for Outstanding New Faculty for the 2010-2011 academic year for my commitment to the research, teaching, and service missions of ETSU.

Graph theory is an area of mathematics that models various relationships on discrete sets. These relationships are usually represented by "vertices," which represent the individuals in the set, and "edges," which describe the various relations between individuals. Applications of graph theory include computer networks, airport connections, and communication arrays. Most of my work has focused on three specific areas of graph theory. The first is problems relating to edge decompositions of graphs and the properties of the intersection graph generated by the decomposition. The second is games on graphs, in particular peg solitaire on graphs. The third deals with pebbling and rubbling, which are inspired by transportation problems. My specific interest and contributions to these areas will be discussed in more detail below. Additional details on my work are available in the papers included in my dossier. I will also discuss the importance of research in a teaching institution and how I have been successful in synthesizing the teaching and research missions of ETSU.

Edge Decompositions of Graphs

The first is problems relating to edge decompositions of graphs and the properties of the intersection graph generated by the decomposition. Given a family of graphs $\mathcal{K}$, an edge decomposition, $\mathscr{D}$, of a graph H (called the *host*) is a partition of the edge set of H such that each part of the partition is isomorphic to an element of $\mathcal{K}$. Traditionally, the parts of this partition are called *blocks* and the elements of $\mathcal{K}$ are called *prototypes* or *divisors*. Of particular interest is the case where $\mathcal{K}$ consists of only a single prototype, G, in which case we say that $\mathscr{D}$ is a G-decomposition of H.

Often in graph decompositions, we restrict our attention to the case when the host is the complete graph. The goal in this case is to establish necessary and sufficient conditions for the existence of such a decomposition. This can be extended to the decomposition of directed complete graphs and mixed complete graphs. I have published two papers on the subject of the decomposition of mixed graphs and two papers on directed graphs.

The intersection graph generated by such an decomposition, denoted $I(\mathscr{D})$ has a vertex for each block in the partition and two vertices share a common edge in $I(\mathscr{D})$ if and only if the corresponding blocks share a common vertex in H. If $I(\mathscr{D})$ is isomorphic to H, we say that

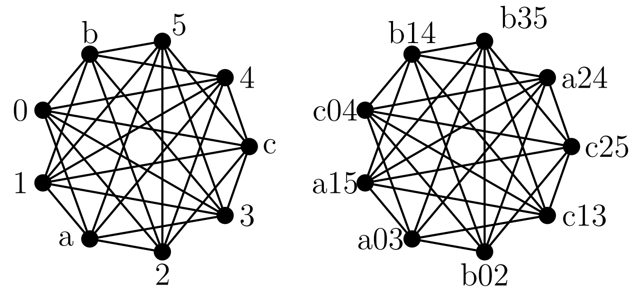

Figure 1: $\overline{K_3} \vee C_6(2,3)$ and a K_3-Decomposition Graph

$\mathscr{D}$ is an *automorphic* G-decomposition of H. An example of an automorphic decomposition and its intersection graph are given in Figure 1.

Most of the known examples of these decompositions are obtained by cyclic decompositions. Informally, a *cyclic* decomposition, is one is which there exists a cyclic permutation g, such that if $A, B \in \mathscr{D}$ are G-blocks of the the decomposition, then the block A may be obtained by applying g^i to B for some $i \in \mathbb{N}$. It is well known that a graceful labeling will induce a cyclic decomposition of a complete graph, and in fact this decomposition is automorphic. The map $f : V(G) \to \{0, ..., |E(G)|\}$ is a *graceful labeling* on G provided that f is injective and for all $uv \in E(G)$, $|f(u) - f(v)| \in \{1, ..., |E(G)|\}$ is unique. Thus my work is connected to several unsolved problems in graph theory, such as Ringel's Conjecture and the Graceful Tree Conjecture. While a graceful labeling is sufficient to induce an automorphic G-decomposition of a complete graph, other related labellings will also work. I have also investigated construction methods for these labelings as a means of obtaining an automorphic decomposition for certain generalizations of the complete graph.

I have also concentrated my research on the necessary conditions for the existence of an automorphic decomposition. These necessary conditions have been used to classify certain graphs which admit an automorphic decomposition. In particular, I have shown that:

(i) If H admits an automorphic P_2-decomposition, then H is the disjoint union of cycles.

(ii) If $\chi(H) = n(G)$, G is d-regular, and H admits an automorphic G decomposition, then H is $n(G)d$-regular.

(iii) If $\chi(H) = n(G)$, G is a p-star, and H admits an automorphic G decomposition, then H is p-regular.

(iv) If G is d-regular and $\mathscr{D}$ is an automorphic G-decomposition of H such that any two G-blocks share at most one common vertex in H, then H is $n(G)d$-regular and G is a complete graph.

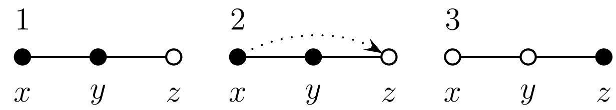

Figure 2: A Typical Jump in Peg Solitaire

Currently, my study of automorphic decompositions has led to three publications. In addition, I have written supplemental material for the Introduction to Graph Theory class that introduces the students to graph decompositions and graceful labelings. This supplement is available on my faculty website.

Peg Solitaire on Graphs

Peg solitaire is a table game which traditionally begins with "pegs" in every space except for one which is left empty (in other words, a "hole"). If in some row or column two adjacent pegs are next to a hole (as in Figure 2), then the peg in x can jump over the peg in y into the hole in z. The peg in y is then removed. The goal is to remove every peg but one. If this is achieved, then the board is considered solved. Perhaps the most famous examples of peg solitaire boards are the 33-hole English cross and the 15-hole triangular board featured in Cracker Barrel restaurants.

There have been a number of papers regarding solving peg solitaire on various boards. My work has generalized this to arbitrary boards which are treated as graphs in the combinatorial sense. A graph is said to be *solvable* if it can be reduced to a single peg. If this is possible regardless of the starting hole, we say that the graph is *freely solvable*. A graph is k*-solvable* if there is a minimum terminal state with k pegs. In particular, if there is a terminal state with two pegs that are distance 2 apart, we say that the graph is *distance 2-solvable*.

Currently, we have given the necessary and sufficient conditions for the solvability of several families of graphs. In particular, we have determined:

(i) The star $K_{1,n}$ is $(n-1)$-solvable; the complete bipartite graph $K_{n,m}$ is freely solvable for $n, m \geq 2$.

(ii) The path P_n is freely solvable if and only if $n = 2$; P_n is solvable iff n is even or $n = 3$; P_n is distance 2-solvable in all other cases.

(iii) The cycle C_n is freely solvable if and only if n is even or $n = 3$; C_n is distance 2-solvable in all other cases.

(iv) The double star $DS(L, R)$ is freely solvable if and only if $L = R$ and $R \neq 1$; $D(L, R)$ is solvable if and only if $L \leq R + 1$; $DS(L, R)$ is distance 2-solvable if and only if $L = R + 2$; $DS(L, R)$ is $(L - R)$-solvable if $L \geq R + 3$.

(v) The Petersen Graph, the Platonic solids, the Archimedean solids, the complete graph, and the n-dimensional hypercube are freely solvable.

(vi) Necessary and sufficient conditions for a tree of diameter four to be solvable.

(vii) The solvability of all connected non-isomorphic graphs on seven vertices or less.

In addition, we have determined the solvability for a number of classes of caterpillars and a generalization of the double star. Part of my research program is to continue this classification for additional families of graphs. Two ways that I intend to do this include determining the solvability of trees of diameter five as well as extending the characterization to other classes of caterpillars.

We have also determined several methods to construct new solvable graphs from others. Perhaps the most striking of these results deals with the Cartesian product of two graphs. We have shown that the Cartesian product of two solvable graphs is likewise solvable, the Cartesian product of a solvable graph and a distance 2-solvable graph is solvable, and the product of two distance 2-solvable graphs is solvable.

Another aspect of this problem is the *fool's solitaire variant*. In this variant, we strive to maximize the number of pegs left at the end of game under the caveat that we make a jump whenever one is available. We define this maximum number of pegs to be the *fool's solitaire number*. We have proven several upper bounds for the fool's solitaire number. The upper bounds were instrumental in determining the fool's solitaire number of several families of graphs.

Recently, I published a paper in *Journal of Combinatorial Mathematics and Combinatorial Computing* that considers the two-player variant known as *duotaire*. In duotaire, the first player selects the initial hole. The players then alternate making peg solitaire jumps on the graph. When no jumps are available, the last player to remove a peg wins. We determined explicit winning strategies for the complete graph, complete bipartite graph, double star, path, and cycle. In this same paper, we considered a variation of duotaire in which one player tries to maximize the number of pegs left at the end of the game whereas their opponent tries to minimize this number. When both players make optimal moves, the result is a fixed parameter. We determined these numbers for complete graph, complete bipartite graph, double star, path, and cycle.

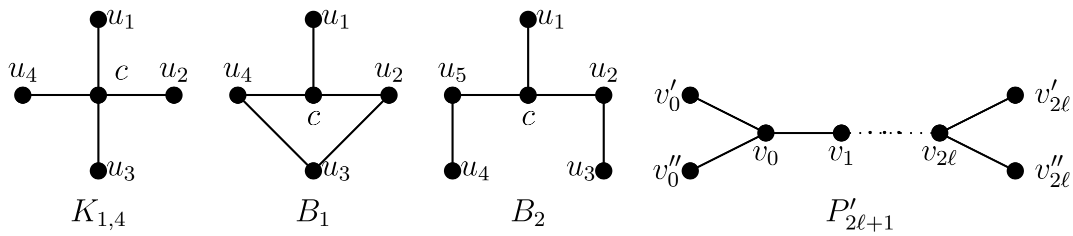

Figure 3: Examples of graphs in which Player One has a winning strategy

The study of peg solitaire has proven to be a fruitful line of research for both me and my students. Currently, my research on peg solitaire has yielded eleven papers accepted in peer reviewed journals. Other researchers have cited our work and considered additional variations. We have also presented talks at Clemson Univerity, the Southeastern International Conference on Combinatorics, Graph Theory, and Combinatorics in Boca Raton, the AMS Special Session in Huntsville, Alabama, and the Cumberland Conference on Combinatorics, Graph Theory, and Computing. Most notably, in June 2022 there was an "Online symposium for Peg Solitaire on Graphs" in which I gave an invited talk "Peg Solitaire on Graphs: A Retrospective with Open Problems". I am confident that the study of peg solitaire on graphs will yield additional scholarly publications in the future.

My study of peg solitaire has also led me to consider generalizations of other classic games to graphs. Recently, I published a paper called "Tic-Tac-Toe on Graphs" in *Australasian Journal of Combinatorics*. As with the traditional game, two players take turns placing their respective marks on the vertices of a graph G. The first player to place their marks on vertices x, y, and z such that $xy \in E(G)$ and $yz \in E(G)$ wins. In this paper, I prove that the first player has a winning strategy on G if and only if G has one of the graphs illustrated in Figure 3 as a subgraph.

Pebbling and Rubbling

In *pebbling* and *rubbling*, we begin with a distribution of "pebbles" to the vertices of the graph. In pebbling, we allow a move $p(u \to v)$ which removes two pebbles from the vertex u and places one pebble on an adjacent vertex v. In rubbling, we allow an additional move $r(u, w \to v)$. This move removes one pebble from each of u and w and places one pebble on a common neighbor v. Both problems are motivated by transportation problems in which we lose units of fuel in order to get to our destination. There are numerous variations of both problems. However, these variations can be broadly partitioned into two variations.

In the first variation, the player gives k pebbles to a "pebbler" who distributes those pebbles amongst the vertices in any way that they wish. The goal is to find the minimum k

so that the player can always move a pebble to any vertex in the graph. Dr. Teresa Haynes, Dr. Rodney Keaton, and I are finishing a paper in which the player tries to place one pebble on every vertex of a dominating set. In other words, the player wants every vertex to either have a pebble or be adjacent to a vertex with a pebble. Dr. Haynes will attest to the fact that there are literally dozens of variations on graph dominations. This being the case, we could use each of these variations as a basis for the set we wish to cover. In particular, we are currently working with a Master's student on a variation based on Roman domination[1]. We also plan to do a variation based on total domination with Dr. Mike Henning.

In the second variation, the player gets to choose the initial configuration of k pebbles. Again, the player seeks the minimum k and a configuration of k pebbles so that they can move a pebble to any vertex on the graph. Dr. Haynes and I worked with a Master's student on a variation in which the player is restricted to placing at most t pebbles on any vertex initially. This variation has yielded two publications in *Congressus Numerantium* and the highly selective *Discussiones Mathematicae Graph Theory*, which boasts an eighty percent rejection rate.

Other Research

My research in graph theory has also lent itself to several exciting student collaborations in addition to those listed above. One of the more fruitful avenues of student research is to assign them a graph and ask them to find out all that they can about the graph. In particular, students are asked to determine several properties of the graph including the order of the graph, the size of the graph, Hamiltonicity, vertex chromatic numbers, edge chromatic numbers. I have also found that my work on peg solitaire is very appealing to students.

This research is related to several classic problems in graph theory and combinatorial designs. As previously mentioned, the labelings on graphs that will induce and automorphic decomposition are related to graceful labelings. It is also possible to obtain these decompositions through the existence of projective planes. As such, my work is related to the Prime Power Conjecture and the existence of perfect difference sets.

In addition to my specialization in edge decompositions and games on graphs, I am also interested in several areas of discrete mathematics. In addition to the combinatorial arguments used to prove certain graph theoretical results, I am interested in formal power

[1]Dr. Haynes, Dr. Steve Hedetniemi, and I also published a paper on a variation of Roman domination in the prestigious *Discrete Applied Mathematics*

series, as well as their application to generating functions. Several of the aforementioned labelings on graphs arise from groups. I have also used several classical results from number theory, such as Bertrand's Postulate and the Frobenius number to prove certain results. Further, results in classic peg solitaire are often proven by defining an algebraic structure. This exposure to number theory and abstract algebra has allowed me to teach our Elementary Number Theory and Introduction to Modern Algebra course, despite not being a number theorist or an algebraist. I am also interested in several areas of general mathematical interest, such as the History and Philosophy of Mathematics. My interest in the History of Mathematics has led to a talk on the Enigma ciphering machine used by the Germans in World War II and a paper on Polish Mathematics published in *The Mathematical Scientist*. The abstracts of my talks are included in my dossier.

Through my studies, I have grown as a mathematician and as a teacher. Many of the proof techniques used in my research are difficult, relying on several different areas of mathematics. I find that this gives me a great sense of perspective to be working on something that is as difficult for me as calculus is for my students. It is important for students to understand that mathematics is a growing field, with discovery taking place through research. I also find that by researching the history of mathematics, I can give my students a better motivation for the material in class. It is also important to keep abreast of ongoing research in mathematical education, and incorporate new techniques into the classroom. Further, as an active researcher, I am better able to mentor my students. This allows me to help them to take their first exciting steps into our field.