Group Theory and the Rubik's Cube

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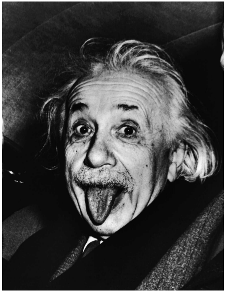
Word of the day

Metagrobology - fancy word for the study of puzzles.



"If you can't explain it simply, you don't understand it well enough."
- Albert Einstein

Some Definitions From Group Theory

The *nth symmetric group* is the set of all permutations on $[n]$. The binary operation on this group is function composition.

The *nth alternating group* is the set of all even permutations on $[n]$. The binary operation on this group is function composition. This group is denoted A_n.

The *nth cyclic group* is the set of all permutations on $[n]$ that are generated by a single element a such that $a^n = e$. This group is denoted $\mathbb{Z}_n$.

Some Conventions

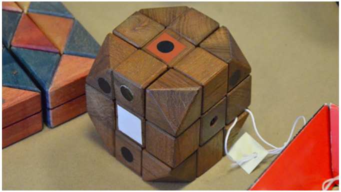
Any permutation will be written as the product of disjoint cycles with fixed points omitted.

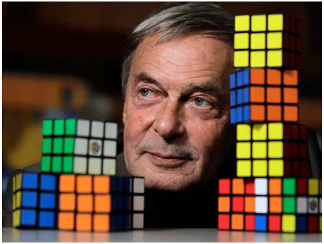
Note that a cycle of odd length is an *even* permutation. Likewise, a cycle of even length is an *odd* permutation.

We will multiply permutations from **left to right**. This matches the literature on the Rubik's Cube, but differs from Fraleigh.

Fun Facts About the Rubik's Cube

Ernő Rubik, a Hungarian professor of architecture invented the Bűvös Kocko (literally, Magic Cube) in 1974.





It was marketed as the Hungarian Magic Cube from 1977 to 1980. From 1980 onwards it was marketed as the Rubik's Cube.

Fun Facts About the Rubik's Cube (part 2)

Over 350 million Rubik's Cubes have been sold. It is unknown how many "knock-offs" or related merchandise have been sold.

Most cubes can be solved in under seventeen moves. Even the most scrambled cube can be solved in at most twenty moves (God's Number).

Fun Facts About the Rubik's Cube (part 3)

The current world record for solving a Rubik's Cube is 3.47 seconds. This was set in November 2018 by Yusheng Du.

Video - Former World Record Solve

Fun Facts About the Rubik's Cube (part 4)

The Rubik's Cube was part of the Hungarian Pavilion during the 1982 World's Fair. However, this sculpture only rotated along one axis.

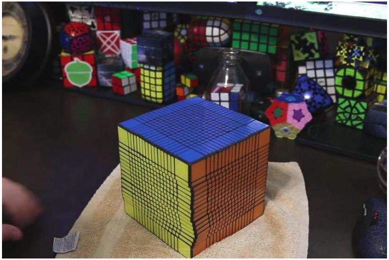
Fun Facts about the Rubik's Cube (Part 5)

The Guinness World Record for the largest fully functional Rubik's Cube was set by Tony Fisher of the United Kingdom. Each side of the cube is 1.57 meters (5 foot, 1.7 inches) long. The cube weighs over 100 kilograms (220 pounds).

Video - Fisher and his giant cube

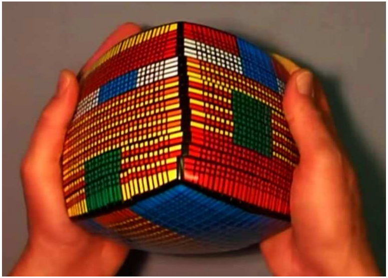
Fun Facts about the Rubik's Cube (Part 6)

The current world record for the largest Rubik's cube (by number of pieces) is the $17 \times 17 \times 17$ cube. It costs over a thousand dollars.



Fun Facts about the Rubik's Cube (Part 7)

However, a Taiwanese company is currently working on a $28 \times 28 \times 28$ cube.



Fun Facts About the Rubik's Cube (part 8)

The most expensive Rubik's Cube was made in 1995 to commemorate the fifteenth anniversary. It has 185 carats of precious gems set in 18-karat gold. It is valued between 1.5 and 2.5 million dollars.

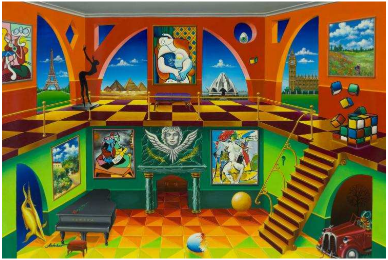
Fun Facts about the Rubik's Cube (Part 9)

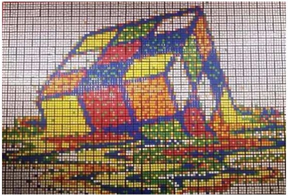
The Rubik's Cube also features in magician Steven Brundage's act. Brundage has been featured on Season 11 of *America's Got Talent* and Season 2 of *Penn & Teller: Fool Us*.

Brundage's Website
Video - Brundage on *Fool Us*

Fun Facts about the Rubik's Cube (Part 10)

The Rubik's Cube has also been featured in art work. People even make mosaics using multiple cubes!





The Rubik's Cube Group

The main objective of this talk is to explore the group of permutations on the Rubik's Cube.

The Rubik's Cube Group (Part 2)

This group of permutations is creatively called *the Rubik's Cube Group*. This group will be denoted $\mathcal{R}$.

In doing so, we will use the concrete example of the Rubik's Cube to illustrate several abstract concepts in group theory.

Generators of the Rubik's Cube

Like many permutation groups, there are many possible generating sets for the Rubik's Cube.

One of the most natural sets is $\{R, L, U, D, F, B\}$, where R represents a clockwise rotation of the right face and so on.

Generators of the Rubik's Cube (part 2)

Another natural generating set is:

(i) The vertical rotation of the entire cube, ν

(ii) The horizontal rotation of the entire cube, ρ

(iii) The clockwise rotation of a single face, say U.

The advantage of this set of generators is that it generalizes nicely to the $n \times n \times n$ cube.

Generators of the Rubik's Cube (part 2)

A more useful set of generators is:

(i) ν, ρ, U

(ii) $R^{-1}UF^{-1}U^{-1}$

(iii) $R^{-1}D^{-1}RD$

(iv) $(URU^{-1}R^{-1})(U^{-1}F^{-1}UF)$, $(U^{-1}F^{-1}UF)(URU^{-1}R^{-1})$

(v) $F(RUR^{-1}U^{-1})F^{-1}$

(vi) $RUR^{-1}URU^2R^{-1}$

(vii) $URU^{-1}L^{-1}UR^{-1}U^{-1}L$

Abelian?

A natural question is whether $\mathcal{R}$ is abelian. That is, do permutations commute inside of $\mathcal{R}$? As an example, consider the permutations UR (on the left) and RU (on the right).

Cubies

To discuss the order of the group, we need to introduce three terms. A *cubie* is any piece of the Rubik's Cube. These are the pieces that get moved around as we play with the puzzle. Naturally, they rest in positions called *cubicles*. A *facelet* is one face of a cubie.

There are 6 center cubies, 8 corner cubies, and 12 edge cubies. Note that cubies can only exchange with other cubies of the same type. Further, center cubies can only rotate in place.

A Little Combinatorics

An upper bound on the order can be computed as follows:

(i) There are 8! ways to permute the corners.

(ii) There are 3^8 ways to rotate the corners.

(iii) There are 12! ways to permute the edges.

(iv) There are 2^{12} ways to flip the edges.

This gives an upper bound of $2^{12} * 3^8 * 8! * 12!$. However, not all of the above actions are possible...

Useful Notation

Singmaster introduced a useful notation for describing the cubies which is independent of the colors used on the cube.

For example, uf is the edge cubie in the up, front position.

Similarly, dbl is the corner cubie in the down, back, left position.

Useful Notation (part 2)

When writing down permutations of cubies, the order of the letters DOES matter. This order represents the orientation of the cubies.

Consider the 8-cycle $(ur, uf, ul, ub, ru, fu, lu, bu)$. While this is an 8-cycle, it only involves four cubies. The effect of this cycle is that after four iterations, each cubie has returned to its original position. However, each of these cubies has been flipped - A Möbius trip!

Useful Notation (part 3)

Thus, we could represent $(ur, uf, ul, ub, ru, fu, lu, bu)$ more compactly as the "flipped 4-cycle" $(ur, uf, ul, ub)_+$. Here, the '$+$' indicates that flipping has occurred.

We will use a similar notation for the rotation of corner cubies. Namely, we will use '$+$' to indicate clockwise rotation and '-' to indicate counter-clockwise rotation.

Which Permutations are Impossible?

Consider any of the face turns, say U. This can be represented a product of two disjoint 4-cycles. The first 4-cycle is a permutation on the edges. The second is a permutation on the corners. Thus,

$$U = (uf, ul, ub, ur)(ufl, ulb, ubr, urf).$$

Note that 4-cycles are odd permutations. The product of two odd permutations is an even permutation.

Ergo, *ANY* permutation on the cube is an *EVEN* permutation!

Which Permutations are Impossible? (part 2)

Note that there are two ways to get an even permutation. An even permutation is either:

(i) A product of two even permutations.

(ii) A product of two odd permutations.

Further, any permutation on the cube is a product of a permutation on the edges and a permutation on the corners.

Which Permutations are Impossible? (part 3)

It follows from the above discussion that either:

(i) The permutation on the edges is an even permutation and the permutation on the corners is an even permutation.

(ii) The permutation on the edges is an odd permutation and the permutation on the corners is an odd permutation.

The upshot of this is that at most half of the potential permutations are possible!

Twists and Flips

Before discussing orientation of cubies, we first define the *chief facelet* of each edge and corner cubie. If a cubie is in the Up (Down) layer, then the Up (Down) facelet will be the chief facelet. If a cubie is in the middle layer, then the Right (Left) facelet will be the chief face.

So, no matter how the cube is scrambled the chief facelet of each cubie never changes.

When the cube is scrambled, the chief facelet may change orientation with respect to the chief facelet of the cubicle it occupies.

Twisted and Flips (part 2)

So if $ur \mapsto rf$, then the edge piece ur has not been *flipped*. However, if $ur \mapsto fr$, then the edge piece has been flipped.

A similar convention exists for corner cubies:

(i) If $urf \mapsto dlb$, then the corner is not *twisted*.

(ii) If $urf \mapsto rub$, then the corner is twisted clockwise.

(iii) If $urf \mapsto ldf$, then the corner is twisted counter-clockwise.

Flipping Out

Let's examine how a quarter turn of each of the six faces affects the orientation of the edge cubies:

(i) If the Up (Down) face is turned, then no piece leaves the respective layer. Thus, no piece changes orientation.

(ii) If the Front (Back) face is turned, then the chief face of every edge cubie is placed in the chief face of an edge cubicle. Thus no edge piece changes orientation.

(iii) If the Right (Left) face is turned, then all four edge cubies are flipped.

So, the total number of flips must be an even number!

Flipping Out (part 2)

The above argument would seem to suggest that the number of flips must be divisible by four. Consider the move RUR^{-1}. This will affect the orientation of five edge cubies ur, rf, rb, dr, and ub as follows:

$$
\begin{array}{ccccccc}
 & R & & U & & R^{-1} & \\
\hline
ur & \mapsto_+ & br_+ & \mapsto & br_+ & \mapsto_+ & ur \\
rf & \mapsto_+ & ru_+ & \mapsto & fu_+ & \mapsto & fu_+ \\
rb & \mapsto_+ & rd_+ & \mapsto & rd_+ & \mapsto_+ & rb \\
dr & \mapsto_+ & fr_+ & \mapsto & fr_+ & \mapsto_+ & dr \\
ub & \mapsto & ub & \mapsto & ur & \mapsto_+ & fr_+
\end{array}
$$

So, the number of flips must be an even number, though not necessarily one that is divisible by four. Hence, half of the potential orientations are possible.

A Twisted Sort of Logic

Let's examine how a quarter turn of each of the six faces affects the orientation of the corner cubies:

(i) If the Up or Down face is turned, no piece leaves the respective layer. Thus, no piece changes orientation.

(ii) If the Front face is turned, then two of the pieces, *ufl* and *frd*, get twisted clockwise. Likewise, *urf* and *fdl* get twisted counter-clockwise. So, a net change in orientation of zero.

(iii) If the Right face is turned, *urf* and *drb* are rotated clockwise. Likewise, *drf* and *urb* are rotated counter-clockwise. Again, the net change in orientation of zero.

So, if we think of a clockwise rotation as a $+1/3$ "charge" and a counter-clockwise twist as a -$1/3$ "charge," then the "total charge" of the must be an integer!

A Twisted Sort of Logic (part 2)

The above argument would suggest that the total charge is zero. However, consider the move RUR. This affects the charge on the corner cubies urf, urb, drb, drf, and ulf as follows:

$$
\begin{array}{ccccccc}
 & R & & U & & R & \\
\hline
urf & \mapsto_+ & bru_+ & \mapsto & lbu_+ & \mapsto & lbu_+ \\
urf & \mapsto_- & brd_- & \mapsto & brd_- & \mapsto_+ & drf \\
drb & \mapsto_+ & frd_+ & \mapsto & frd_+ & \mapsto_- & urf \\
drf & \mapsto_- & fru_- & \mapsto & rbu_- & \mapsto_- & rdb_+ \\
ulf & \mapsto & ulf & \mapsto & ufr & \mapsto_+ & bur_+
\end{array}
$$

Notice that the total charge here is $+1$.

Quarks, Mesons, and Baryons (oh my)

By the above discussion, the rotation of a single corner is impossible. Since corners can have a $1/3$ rotation (think $1/3$ charge), we call them *quarks*.

Much like the quarks from particle physics, quarks are never isolated.

A quark may be accompanied by a corner rotated in the opposite direction (an *anti-quark*). This quark/anti-quark pair is called a *meson*.

Otherwise, there must be three corners all rotated in the same direction. This set of three quarks is called a *baryon*.

Quarks, Mesons, and Baryons (part 2)

Quarks only exist as part of a meson or a baryon.

In either case, the "total charge" of the cube must be an integer. Hence, only one third of the potential corner rotations are possible.

Generating the Rubik's Cube (again)

We want to show that the remaining permutations are all possible. To do this, we must show that each of the following is possible:

(i) an arbitrary double edge-pair swap,
(ii) an arbitrary double corner-pair swap,
(iii) an arbitrary two-edge flip,
(iv) an arbitrary meson.

Note that any three-element permutation (i.e., a baryon) can be obtained by overlapping the elements above.

The Power of Congugacy

Showing that any *arbitrary* element from each of the above types is achievable would be a lot of work. Fortunately, we need only show that a *single* element of each type is possible. Why?

Recall that if $a, g \in G$, then the *conjugation* of a by g is the element gag^{-1}.

The Power of Congugacy (part 2)

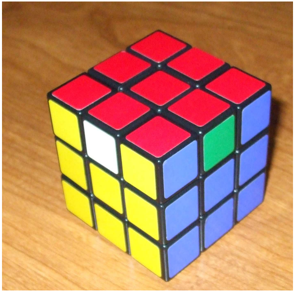
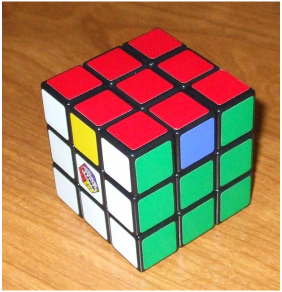
Suppose that we can flip an opposite pair of edges on the top face (say *uf* and *ub*) while leaving the other cubies fixed. We denote this permutation as ξ.

Further, we want to flip *uf* and *dl* while leaving the other cubies fixed. We move the down left edge to the top back position with the moves $D^{-1}F^2$. Then apply ξ. Now we apply $(D^{-1}F^2)^{-1} = F^2D$ to return the (now flipped) down left edge to its original position.

So, we have conjugated ξ by $D^{-1}F^2$. This is represented by:

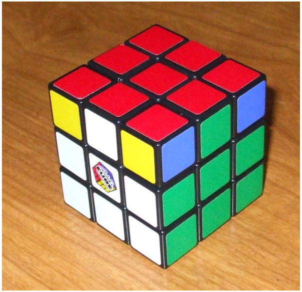
$$(D^{-1}F^2)\xi(D^{-1}F^2)^{-1}.$$

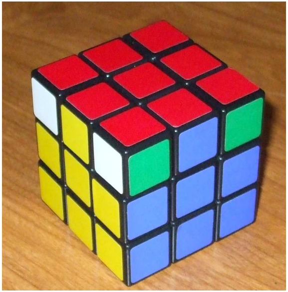
Double Edge-pair Swap

$$(R^2F^2B^2L^2)D(R^2F^2B^2L^2)U$$

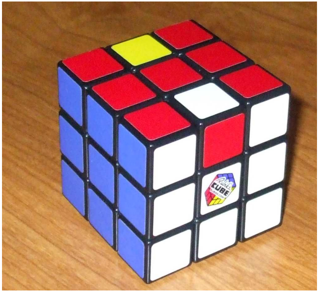
Double Corner-pair Swap

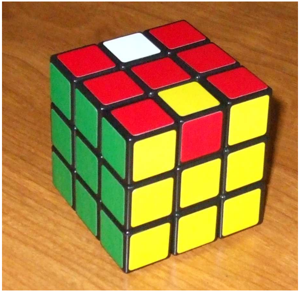




$$(R^2F^2B^2L^2)D(R^2F^2B^2L^2)U^{-1}$$

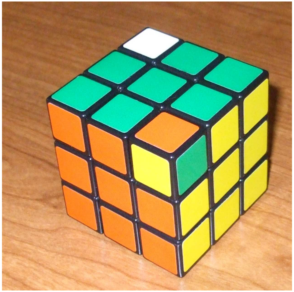
Two-edge Flip

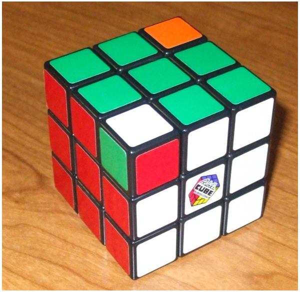




$$(FU^{-1}RF^{-1}U)(RL^{-1})(B^{-1}UR^{-1}BU^{-1})(R^{-1}L)$$

A Meson





$$\left((BR^{-1}D^2RB^{-1})U^2\right)^2$$

The Order of $\mathcal{R}$

We can obtain all of the non-excluded permutations. Thus, the order of $\mathcal{R}$ is

$$\frac{2^{12} * 3^8 * 8! * 12!}{2 * 2 * 3} \approx 4.3 \times 10^{19}$$

Commutators and Conjugates

You may notice that we have written many of our elements with terms such as $xyx^{-1}y^{-1}$ or xyx^{-1}.

Recall that xyx^{-1} is the conjugate of y by x. Think of this as a "set-up" to do permutation y.

The element $xyx^{-1}y^{-1}$ is the *commutator* of x and y. Often these moves do the "heavy lifting" in our solution.

Speed-cubers use a combination of conjugates and commutators to rapidly progress through the solution.

The Chinese Remainder Theorem

What is the order of

$$RU = (ur, br, dr, fr, uf, ul, ub)(urf)_+(ubr, bdr, dfr, luf, bul)_-?$$

This is the product of a 7-cycle, a 3-cycle, and a 15-cycle. The *Chinese Remainder Theorem* states that the order of this element is *least common multiple* of the cycle lengths. In this case, $lcm(3, 7, 15) = 105$.

The Chinese Remainder Theorem (part 2)

Incidently, here are the orders of some other elements:

(i) $o(F^2B^2R^2L^2U^2D^2) = 2$
(ii) $o(RL^{-1}BF^{-1}DU^{-1}RL^{-1}) = 3$
(iii) $o(FRF^{-1}R^{-1}) = 6$.
(iv) $o(RU) = 105$.
(v) $o(FL^2R^{-1}UR^{-1}) = 1260$.

It turns out that 1260 is the order of the largest cyclic subgroup of $\mathcal{R}$. What do other subgroups look like?

Subgroups

Recall that H is a *subgroup* of a group G is H forms a group under the binary operation associated with G.

There are many subgroups associated with $\mathcal{R}$. Cyclic subgroups are of course all over the place.

One of my favorite subgroups associated with $\mathcal{R}$ is the *slice group*. This is the subgroup generated by $\{RL^{-1}, FB^{-1}, UD^{-1}\}$. Similarly, the *anti-slice group* is generated by $\{RL, FB, UD\}$. These are both non-abelian.

A Non-cyclic Abelian Subgroup

Consider the group generated by $\{R^2L^2, F^2B^2, U^2D^2\}$. This subgroup has eight elements:

$$e, R^2L^2, F^2B^2, U^2D^2, (R^2L^2)(F^2B^2),$$

$$(R^2L^2)(U^2D^2), (F^2B^2)(U^2D^2), (R^2L^2)(F^2B^2)(U^2D^2).$$

Up to isomorphism, there are only five groups of order eight. Hence, it is easy to see that this subgroup is isomorphic to $\mathbb{Z}_2^3$.

Normal Subgroups

Recall that N is a *normal subgroup* of G if $\forall g \in G$ and $\forall n \in N$, $gng^{-1} \in N$.

Consider the subgroup of $\mathcal{R}$ that leaves all corner cubies fixed. In other words, this subgroup consists of all permutations and flips of edge cubies. It is easy to see that this group is normal in $\mathcal{R}$. Call this subgroup $\mathcal{N}$.

Cosets

The *cosets* of $\mathcal{N}$ are of the form $g\mathcal{N}$, where $g \in \mathcal{R}$.

A permutation that affects only the corners is disjoint from one that affects only the edges. Hence, these permutations commute. From this it follows that the cosets of $\mathcal{N}$ are of the form $g\mathcal{N}$, where g is a permutation/rotation on the corners.

Factor Groups

Recall that if N is a normal subgroup of G, then the *factor group* G/N is the group on the cosets of N where multiplication is defined by $(aN)(bN) = (ab)N$.

In the case of $\mathcal{R}/\mathcal{N}$, we can essentially think of the edge/center cubies as having no impact on the structure. Hence, we are "modding out" by the edge permutations. For this reason, only permutations and rotations of the corner cubies matter.

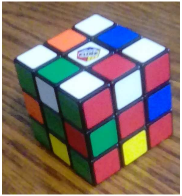
Hence, $\mathcal{R}/\mathcal{N}$ is (basically) isomorphic to the group of permutations on a $2 \times 2 \times 2$ Rubik's Cube.

The Center Subgroup

Recall that the *center* of a group is the set of elements of the group that commute with every other element. In other words,

$$Z(G) = \{x \in G : xg = gx, \forall g \in G\}.$$

Note that the center is always a normal subgroup.
In the case of the Rubik's Cube Group, the center is the identity and the *superflip*. The superflip is the permutation that leaves all cubies in their original position, but flips all of the edge pieces.



$$UR^2FBRB^2RU^2LB^2RU^{-1}D^{-1}R^2FR^{-1}LB^2U^2U^2F^2$$

Another Normal Subgroup

Consider the group of permutations in $\mathcal{R}$ that leaves every cubie in its original position, but alters their orientation. We call this group the *orientation cube group*. We denote it $\mathcal{O}$. Again, $\mathcal{O}$ is a normal subgroup of $\mathcal{R}$.

We'll take a second subgroup of $\mathcal{R}$ to be the subgroup of all permutations that alter the positions of the cubies, but leaves the orientations alone. This group is the *permutation cube group*. Denote this subgroup as $\mathcal{P}$.

A Funny Thing About Those Last Two Subgroups

Note that $\mathcal{O} \cap \mathcal{P} = \{e\}$. Further, $\forall g \in \mathcal{R}$, there is a unique way to represent g as a product $g = ab$, where $a \in \mathcal{O}$ and $b \in \mathcal{P}$.

For this reason, we say that $\mathcal{R}$ is a *semi-direct product* of $\mathcal{O}$ and $\mathcal{P}$. This situation is denoted $\mathcal{R} = \mathcal{O} \rtimes \mathcal{P}$.

We are motivated by the above comments to determine the structure of $\mathcal{O}$ and $\mathcal{P}$.

The Orientation Cube Group

Note that the orientations of the corner cubies is separate from the orientations of the edge cubies.

Each of the twelve edge cubies has two possible orientations. For the reasons discussed above, the orientation of eleven edge cubies will determine the orientation of the twelfth edge cubie. For this reason, this subgroup of permutations is isomorphic to $\mathbb{Z}_2^{11}$.

The Orientation Cube Group (part 2)

Similarly, each of the eight corner cubies has three possible orientations. Since isolated quarks are not possible, the orientation of seven corner cubies will determine the orientation of the last one. Hence, this subgroup is isomorphic to $\mathbb{Z}_3^7$.

Ergo, $\mathcal{O} = \mathbb{Z}_2^{11} \times \mathbb{Z}_3^7$.

The Permutation Cube Group

Again, we think of $\mathcal{P}$ as a product of two subgroups - the group of permutations on the eight corner cubies and the group of permutations on the twelve edge cubies. These permutations must have the same parity. Thus, $A_8 \times A_{12}$ contains exactly half of the permutations in $\mathcal{P}$. For this reason, $A_8 \times A_{12}$ is a normal subgroup of $\mathcal{P}$.

The Permutation Cube Group (part 2)

The permutation on the corners and the permutation on the edges must have the same parity. Further, $A_8 \times A_{12}$ is normal in $\mathcal{P}$. So, we write $\mathcal{P}$ as a semi-direct product of $A_8 \times A_{12}$ and the group $\{e, (ur, uf)(urf, urb)\}$.

If e is chosen from $\{e, (ur, uf)(urf, urb)\}$, then both permutations are even. If $(ur, uf)(urf, urb)$ is chosen, then both permutations are odd. Note that $\{e, (ur, uf)(urf, urb)\}$ is isomorphic to $\mathbb{Z}_2$.

For this reason,

$$\mathcal{P} = (A_8 \times A_{12}) \rtimes \mathbb{Z}_2.$$

The Rubik's Cube Group

So, we have that

$$\mathcal{R} = (\mathbb{Z}_2^{11} \times \mathbb{Z}_3^7) \rtimes ((A_8 \times A_{12}) \rtimes \mathbb{Z}_2).$$

Although, it is usually more convenient (though less precise) to say that $\mathcal{R}$ is a particular subgroup of A_{48}.

What About Cubes with Oriented Centers?

You may have seen "souvenir" cubes which have pictures, rather than colors on each side of the cube. To solve such a cube, the image on the center must line up with the image on the surrounding pieces. In such a case, we say that the centers on the cube are *oriented*.

Oriented Centers (part 2)

How does this affect the permutation group?

Consider any permutation that restores the position center and edge cubies while ignoring the centers. As discussed above, this must be an even permutation. Hence, it must involve an even number of face turns. Ergo, such a permutation will result in an even number of center rotations. Since each of the six centers has four possible rotations, there are 4^6 possible orientations of the centers. However, since the total sum of the rotations must be even, we divide by 2.

This gives us $4^6/2$ possible rotations of the centers.

Oriented Centers (part 3)

To put this another way, five of the centers can have any rotation. However, since the total rotation must be even, the final center is limited to two possible orientations. So if the sum rotation of the first five centers is even, then the final center must be rotated either 0 or 180 degrees. Likewise, if the sum rotation of the first five centers is odd, then the final center must be rotated either 90 or 270 degrees.

So the rotational group of the (oriented) centers is isomorphic to $\mathbb{Z}_4^5 \times \mathbb{Z}_2$.

Oriented Centers (part 4)

Two examples of moves that rotate the centers and restore all other cubies:

(i) Rotate top center 180 degrees:

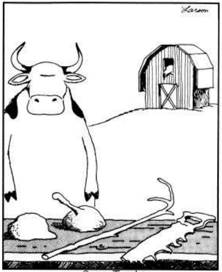
$$(URLU^2R^{-1}L^{-1})^2.$$

(ii) Rotate top center clockwise, right center counter-clockwise:

$$RL^{-1}FB^{-1}UD^{-1}R^{-1}U^{-1}DF^{-1}BR^{-1}LU.$$

Using More Tools...

One way of looking at modern algebra is that it attempts to answer the question "What can we do with a given set of tools?" In this case, we know what permutations are possible with the Rubik's Cube. Note that if we allow additional moves (say flipping one edge), then we can get all of the "forbidden configurations" in which one edge has been flipped.



Cow Tools

Nested Subgroups

The process of creating semi-direct product to describe $\mathcal{R}$ may remind you of the process used to find *solvable groups* in Galois theory or a *composition series* for the Jordan-Hölder Theorem.

In both cases, the goal is to create a nested series of proper subgroups of G:

$$\{e\} = G_0 < G_1 < \cdots < G_{n-1} < G_n = G$$

such that the series $\{G_0, G_1, \ldots, G_n\}$ satisfies certain properties.

Can we use the notion of nested subgroups to aid in solving the Rubik's Cube?

Nested Subgroups (part 2)

In 1981, Morwen Thistlethwaite devised a nested subgroup series for $\mathcal{R}$:

(i) $G_4 = \mathcal{R} =< L, R, F, B, U, D >$

(ii) $G_3 =< L, R, F, B, U^2, D^2 >$

(iii) $G_2 =< L, R, F^2, B^2, U^2, D^2 >$

(iv) $G_1 =< L^2, R^2, F^2, B^2, U^2, D^2 >$

(v) $G_0 = \{e\}$

Nested Subgroups (part 3)

Some notes on Thistlethwaite's algorithm:

(i) $\mathcal{R}$ can be reduced to G_3 in no more than seven moves.

(ii) G_3 can be reduced to G_2 in no more than thirteen moves from G_3.

(iii) G_2 can be reduced to G_1 in no more than fifteen moves from G_2.

(iv) G_1 can be reduced to the identity in no more than seventeen moves from G_1.

Using Thistlethwaite's algorithm at most fifty-two moves are needed to solve the Rubik's Cube. This number has since been reduced down to 45 moves.

A Few Words on the Word Problem

Often times elements of a group can be represented in a number of different ways. For example, in $\mathcal{R}$:

$$D = (RL^{-1}F^2B^2RL^{-1})U(RL^{-1}F^2B^2RL^{-1}).$$

In this case, our "alphabet" is our generating set $\{R, L, F, B, U, D\}$ (and their inverses) and our "words" are anything that can be written as a sequence of those letters.

A Few Words on the Word Problem (part 2)

The most common description of *the word problem* is to decide if two "words" actually represent the same element in the group.

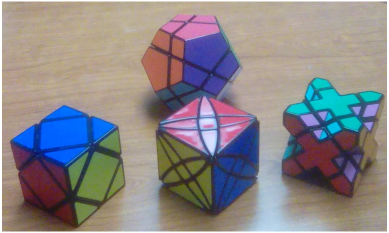
Another version: Given any element in the group and an alphabet, find the shortest representation of that element as a word in that alphabet.

Other Puzzles - The Platonic Solids

Rubik's cube type toys also exist for the other Platonic solids. Sadly, I do not own a Rubik's Icosahedron as it is quite expensive.

Other Puzzles - Skewbs

In a Skewb puzzle, the corners rotate. This moves the adjacent centers and the adjacent corners.

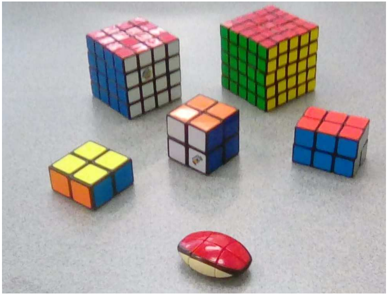


Other Puzzles - About Face?

I have several puzzles that I would classify as "shapeshifters" - the puzzle changes shape as you play with it. In each of these cases, the puzzle has the same internal mechanism as the Rubik's Cube. However, how we should think about the faces is significantly different based on the arrangement.

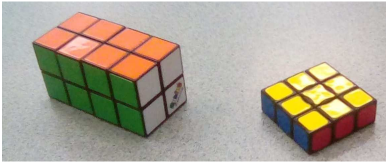
Other Puzzles - Rubik's-type toys with different dimensions

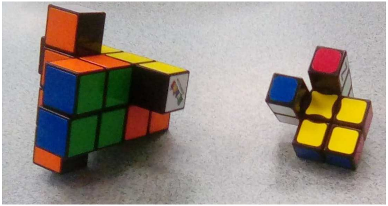
In my collection, I have several variations on a Rubik's cube. Each of these puzzles a different number of cubies along their side. The puzzle group for these puzzles can be determined in a similar manner to the Rubik's cube.



Other Puzzles - Not quite Rubik's-type toys

While the $1 \times 3 \times 3$ and $2 \times 2 \times 4$ puzzle may seem like the Rubik's cube, their moves may result in a configuration of a different shape. Hence the set of their complete state space cannot be analyzed using the earlier techniques.





Other Puzzle - The Molecube

The Molecube has a Rubik's Cube mechanism. However, the pieces have no orientation. Further, two of the corners are green, three of the edges are red, and three of the edges are purple. The goal of this puzzle is to arrange the balls so that each of the nine colors appears exactly once on each face of the cube.

An Open Problem

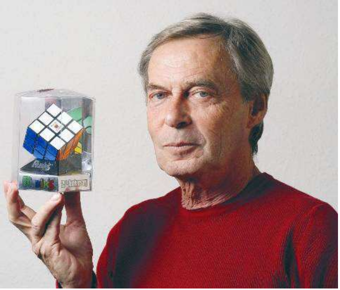
What are ways to mathematically quantify the difficulty of a puzzle such as the Rubik's Cube?

(i) Number of states?
(ii) Word length?
(iii) Psychological problems?
(iv) Similarity to other puzzles.
(v) The Egg of Columbus.

Words of Wisdom from the Master

"If you are curious, you'll find the puzzles around you. If you are determined, you will solve them."

- Ernő Rubik



Additional Reading

(i) Christoph Bandelow. *Inside Rubik's Cube and Beyond.*

(ii) John Ewing and Czes Kośniowski. *Puzzle it Out: Cubes, Groups, and Puzzles.*

(iii) John B. Fraleigh. *A First Course in Abstract Algebra.*

(iv) Alexander H. Frey, Jr. and David Singmaster. *Handbook of Cubik Math.*

(v) Douglas R. Hofstadter. "Magic Cubology" in *Metamagical Themas: Questing for the Essence of Mind and Pattern.*

(vi) David Joyner. *Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and other Mathematical Toys.*

(vii) Ernő Rubik, et al. *Rubik's Cubic Compendium.*

(viii) Jerry Slocum, et al. *The Cube: The Ultimate Guide to the World's Bestselling Puzzle.*

Questions?