Tic-Tac-Toe on Graphs

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Tic-Tac-Toe is a two player pencil and paper game traditionally played on a three by three grid since at least 1884. Players alternate turns placing marks on the grid.

The first player to have three of their respective marks in a horizontal, vertical, or diagonal row wins the game.
Introduction

In the traditional game, perfect play from both players will result in a draw each time.

In addition to the simplicity of the game, this makes Tic-Tac-Toe ideal for teaching children the basics of strategic play.

However, generalizations of the game are much more complicated and often unsolved. See Beck [1] for more information on variations of Tic-Tac-Toe.
Generalizing to Graphs

As with the traditional game, the players take turns placing their respective marks on the vertices of a graph $G$.

The first player to place their marks on vertices $x$, $y$, and $z$ such that $xy \in E(G)$ and $yz \in E(G)$ wins.
A *strategy* is one of the options available to a player where the outcome depends not only on the player’s actions, but the actions of others.

A strategy is *winning* if the player following it must necessarily win, regardless of the actions of their opponents.

A *drawing strategy* is one in which the player following it will force a draw, no matter the actions of their opponents.
Some Useful Notes

The Fundamental Theorem of Combinatorial Game Theory (see for example Siegel [3] and Wells [4]) states that in games such as Tic-Tac-Toe either one player has a winning strategy or both players have a drawing strategy.

Per Williams [6], we can assume that both players will play perfectly and will play for the best possible outcome that they can achieve safely.
In this talk, we will give simple necessary and sufficient conditions for Player One to have a winning strategy on a graph $G$.

These conditions will be a forbidden subgraph characterization along the lines of Beineke’s Theorem or Kuratowski’s Theorem (see West [5] for other examples).

We will show that both players have a drawing strategy on all remaining graphs. Explicit strategies will be provided for all graphs.
A *fork* is a configuration in which one player will win on the next turn, regardless of the actions of the second player.

\[
\begin{array}{cc}
X & O \\
X & X \\
X & O \\
\end{array}
\]
As in the traditional game, executing and defending against forks is central to the strategy of Tic-Tac-Toe on graphs. There are four forks that are possible:

- $K_{1,3}$-fork
- $P_4$-fork
- $C_4$-fork
- $P_5$-fork
Graphs with $\Delta(G) \geq 4$

**Result 1** If $G$ has the star $K_{1,4}$ as a subgraph, then Player One has a winning strategy on $G$. 

![Graph](image-url)
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Result 2 If the maximum degree of $G$ is at most 2, then both players have a drawing strategy on $G$. 
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Graph: 
```
  X   O   X
  |   |   |
  O---O---O
  |   |   |
  X   O   X
```
Result 3 Suppose that $G$ has $H_1$ or $H_2$ (given below) as a subgraph. It follows that Player One has a winning strategy on $G$. 

$H_1$ 

$H_2$
Result 3  Suppose that $G$ has $H_1$ or $H_2$ (given below) as a subgraph. It follows that Player One has a winning strategy on $G$. 

![Graph $H_1$](image1)

![Graph $H_2$](image2)
Result 3 Suppose that $G$ has $H_1$ or $H_2$ (given below) as a subgraph. It follows that Player One has a winning strategy on $G$. 

$H_1$ 

$H_2$
Result 3 Suppose that $G$ has $H_1$ or $H_2$ (given below) as a subgraph. It follows that Player One has a winning strategy on $G$. 
At this point, we need only consider graphs $G$ such that the maximum degree is three and the graph has neither $H_1$ nor $H_2$ as a subgraph. The possibilities for such a graph are:
The “Easy Case”

**Result 4** Both players have a drawing strategy on the first graph on the previous slide.
The “Easy Case”

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\[
\begin{align*}
&X & & \circ \\
&\circ & & \circ \\
&\circ & & \circ \\
\end{align*}
\]
As usual, let $P_n$ denote the graph on the vertices $v_0, v_1, \ldots, v_{n-1}$.

Let $P(n, 1)$ denote the graph obtained from $P_n$ by appending two pendants to $v_0$.

Likewise, let $P(n, 2)$ denote the graph obtained from $P(n, 1)$ by appending two pendants to $v_{n-1}$. 
The Case with $P(n, 1)$

**Result 5** Both players have a drawing strategy on $P(n, 1)$.

This is similar to the result for $\Delta(G) = 2$, except that when given an option, Player Two takes the neighbor closest to the vertex of degree 3.
The Case with $P(n, 1)$

**Result 5** Both players have a drawing strategy on $P(n, 1)$.

This is similar to the result for $\Delta(G) = 2$, except that when given an option, Player Two takes the neighbor closest to the vertex of degree 3.

![Diagram of a graph with vertices and edges]
The Case with $P(n, 1)$

**Result 5** Both players have a drawing strategy on $P(n, 1)$.

This is similar to the result for $\Delta(G) = 2$, except that when given an option, Player Two takes the neighbor closest to the vertex of degree 3.

![Diagram of a graph with vertices and edges](image)
The Case with $P(2k + 1, 2)$

**Result 6** Player One has a winning strategy on $P(2k + 1, 2)$. 

![Graph Diagram]

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Tic-Tac-Toe on Graphs
The Case with $P(2k + 1, 2)$

**Result 6** Player One has a winning strategy on $P(2k + 1, 2)$. 

![Graph Diagram](image-url)
The Case with $P(2k + 1, 2)$

Result 6 Player One has a winning strategy on $P(2k + 1, 2)$. 

```
  O---X---O
 /      \\     /
|       |     |
|       |     |
\      /     
  O---X---O
```
The Case with $P(2k + 1, 2)$

**Result 6** Player One has a winning strategy on $P(2k + 1, 2)$.
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![Graph Diagram]

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**Result 6** Player One has a winning strategy on $P(2k + 1, 2)$. 
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The Case with $P(2k,2)$

**Result 7** Both players have a drawing strategy on $P(2k,2)$.

By the above argument, we can assume that Player Two takes both vertices of degree three and that Player One takes their neighbors of degree two.

Since the path between the X’s is of even length, eventually we reach a situation like in the figure above.
Our Main Result (Part 2)

Theorem

Player One has a winning strategy on $G$ if and only if $G$ has $K_{1,4}$, $H_1$, $H_2$, or $P(2k + 1, 2)$ as a subgraph. Otherwise both players have a drawing strategy on $G$. 
Two Variations

We also consider two variations.

In the restricted variation, only vertices that form a $K_3$ constitute a winning set.

In the induced variation, only vertices that form an induced $P_3$ constitute a winning set.

In both cases we can provide a forbidden subgraph characterization.
Theorem

Player One has a winning strategy on $G$ in restricted Tic-Tac-Toe if and only if $G$ has $R_1$, $R_2$, $R_3$, $R_4$, or $T(2k + 1, 2)$ (arising from $P(2k + 1, 2)$) as a subgraph. Otherwise both players have a drawing strategy on $G$. 

$R_1$  \hspace{1cm} $R_2$  \hspace{1cm} $R_3$  \hspace{1cm} $R_4$  \hspace{1cm} $T(5, 2)$
Induced Tic-Tac-Toe on Graphs

Theorem

Player One has a winning strategy on a graph $G$ in induced Tic-Tac-Toe on graphs if and only if $G$ contains one of the following as an induced subgraph: $K_{1,4}$, $H_1$, $H_2$, $I_1$, $I_2$, $I_3$, $I_4$, $I_5$, $I_6$, or any element of $I(2k + 1, 2)$ (arising from $P(2k + 1, 2)$). Otherwise, both players have a drawing strategy on $G$. 

\[
\begin{align*}
I_1 & \quad I_2 & \quad I_3 & \quad I_4 & \quad I_5 & \quad I_6 \\
\end{align*}
\]
The *pie rule* (also known as the swap rule or Nash’s rule from Hex) is a common method for mitigating the advantage of going first [2]. If the pie rule is implemented, then after the first move is made, Player Two has one of two options:

(i) Player Two lets the move stand. Play then proceeds as normal.

(ii) Player Two “takes” that move. Player One then plays as if they were the second player.

What is the set of graphs in which each player has a winning strategy when the pie rule is implemented?
As another way to neutralize Player One’s advantage, we might also consider a variation in which *double moves* are allowed. There are several possibilities for variations in which double moves are allowed:

(i) On each turn, both players place two marks instead of one.

(ii) On Player One’s first turn, they place a single mark. On Player Two’s first turn, they place two marks. Play then proceeds as normal.

(iii) On Player One’s first turn, they place a single mark. On each subsequent turn, both players place two marks.

(iv) Player Two places two marks for every mark placed by Player One.

For each of the above possibilities, determine necessary and sufficient conditions for each player to have a winning strategy on a graph $G$.  

Open Problem - Full Play Convention

Suppose that we allow play to continue after one player has captured a $P_3$ (this is known as full play convention in [1, 3]).

(i) What is the set of graphs in which Player One cannot prevent Player Two from capturing a $P_3$?

(ii) What is the set of graphs in which Player One can prevent Player Two from capturing a $P_3$ only at the expense of capturing their own?

(iii) What is the set of graphs in which Player One can both capture a $P_3$ and prevent Player Two from capturing a $P_3$?
In a *misère* version of Tic-Tac-Toe, the first player to complete a (not necessarily induced) $P_3$ loses. What are the necessary and sufficient conditions for each player to have a winning strategy on the *misère* version of Tic-Tac-Toe on graphs?
What if we were to consider other subgraphs (induced or otherwise) as our winning sets? In particular, if we were to generalize Connect-Four to graphs, then we would likely assume that both players were trying to capture a $P_4$. Likewise, we could generalize this further by assigning each player a family of graphs (which need not be the same for both players). The first player to capture any graph in their respective family wins.
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