Tic-Tac-Toe on Graphs

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Introduction

Tic-Tac-Toe is a two player pencil and paper game traditionally played on a three by three grid since at least 1884. Players alternate turns placing marks on the grid.

The first player to have three of their respective marks in a horizontal, vertical, or diagonal row wins the game.

Introduction

In the traditional game, perfect play from both players will result in a draw each time.

In addition to the simplicity of the game, this makes Tic-Tac-Toe ideal for teaching children the basics of strategic play.

However, generalizations of the game are much more complicated and often unsolved. See Beck [1] for more information on variations of Tic-Tac-Toe.

Generalizing to Graphs

As with the traditional game, the players take turns placing their respective marks on the vertices of a graph G.

The first player to place their marks on vertices x, y, and z such that $xy \in E(G)$ and $yz \in E(G)$ wins.

Some Useful Notes

A *strategy* is one of the options available to a player where the outcome depends not only on the player's actions, but the actions of others.

A strategy is *winning* if the player following it must necessarily win, regardless of the actions of their opponents.

A *drawing strategy* is one in which the player following it will force a draw, no matter the actions of their opponents.

Some Useful Notes

The Fundamental Theorem of Combinatorial Game Theory (see for example Siegel [3] and Wells [4]) states that in games such as Tic-Tac-Toe either one player has a winning strategy or both players have a drawing strategy.

Per Williams [6], we can assume that both players will play perfectly and will play for the best possible outcome that they can achieve safely.

Our Main Result

In this talk, we will give simple necessary and sufficient conditions for Player One to have a winning strategy on a graph G.

These conditions will be a forbidden subgraph characterization along the lines of Beineke's Theorem or Kuratowski's Theorem (see West [5] for other examples).

We will show that both players have a drawing strategy on all remaining graphs. Explicit strategies will be provided for all graphs.

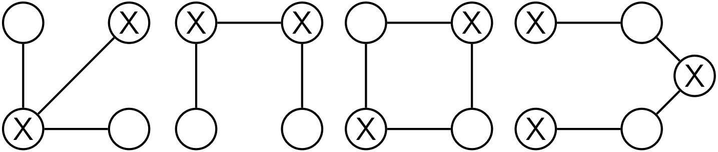
A Few Words About Forks

A *fork* is a configuration in which one player will win on the next turn, regardless of the actions of the second player

X	O	
X	X	
		O

Forks in Tic-Tac-Toe on Graphs

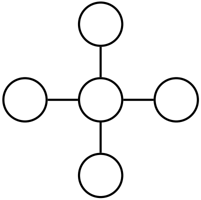
As in the traditional game, executing and defending against forks is central to the strategy of Tic-Tac-Toe on graphs. There are four forks that are possible:



$K_{1,3}$-fork P_4-fork C_4-fork P_5-fork

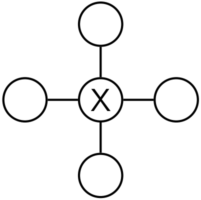
Graphs with $\Delta(G) \geq 4$

Result 1 If G has the star $K_{1,4}$ as a subgraph, then Player One has a winning strategy on G.



Graphs with $\Delta(G) \geq 4$

Result 1 If G has the star $K_{1,4}$ as a subgraph, then Player One has a winning strategy on G.



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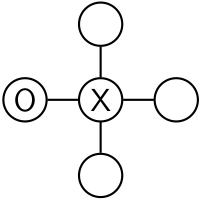

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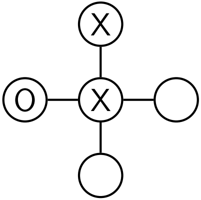
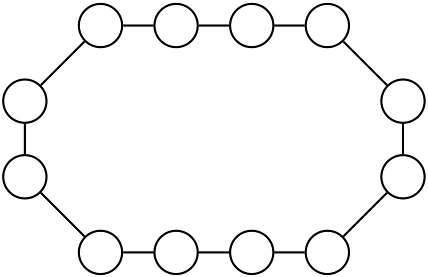


Graphs with $\Delta(G) \leq 2$

Result 2 If the maximum degree of G is at most 2, then both players have a drawing strategy on G.



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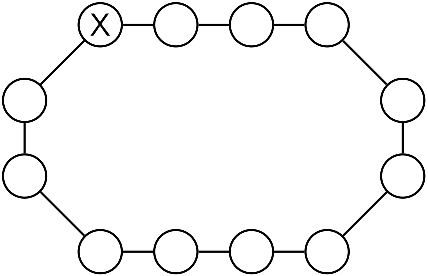

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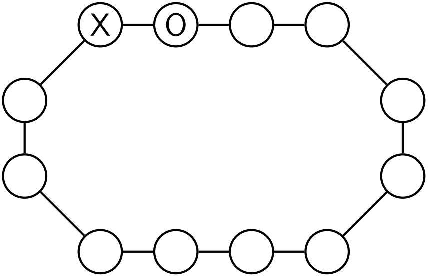

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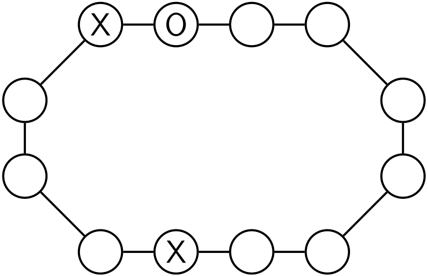

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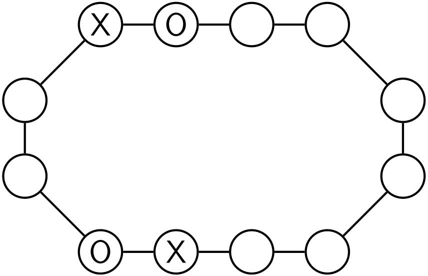
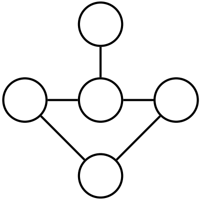
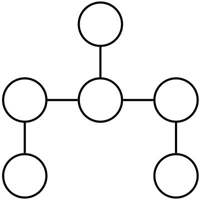


Two Graphs with $\Delta(G) = 3$

Result 3 Suppose that G has H_1 or H_2 (given below) as a subgraph. It follows that Player One has a winning strategy on G.



H_1



H_2

Two Graphs with $\Delta(G) = 3$

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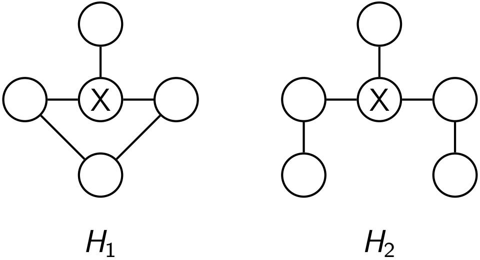

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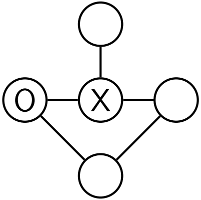

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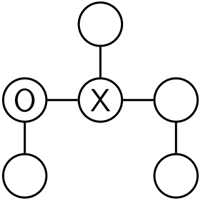

H_2

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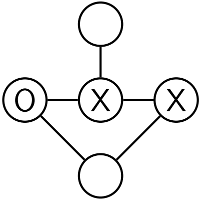

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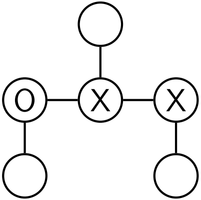
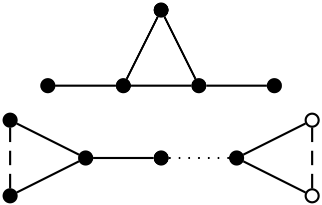


H_2

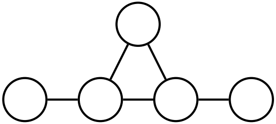
Other Graphs with Maximum Degree 3?

At this point, we need only consider graphs G such that the maximum degree is three and the graph has neither H_1 nor H_2 as a subgraph. The possibilities for such a graph are:



The "Easy Case"

Result 4 Both players have a drawing strategy on the first graph on the previous slide.



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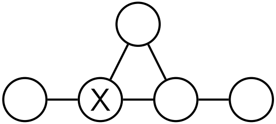

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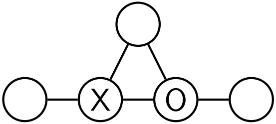


The “Harder Case”

As usual, let P_n denote the graph on the vertices v_0,v_1,...,v_{n-1}.

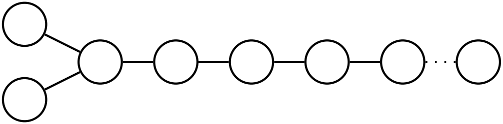
Let $P(n,1)$ denote the graph obtained from P_n by appending two pendants to v_0.

Likewise, let $P(n,2)$ denote the graph obtained from $P(n,1)$ by appending two pendants to v_{n-1}.

The Case with $P(n, 1)$

Result 5 Both players have a drawing strategy on $P(n, 1)$.

This is similar to the result for $\Delta(G) = 2$, except that when given an option, Player Two takes the neighbor closest to the vertex of degree 3.



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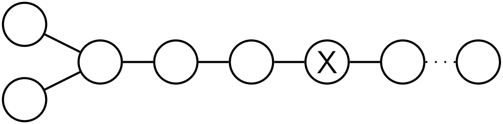

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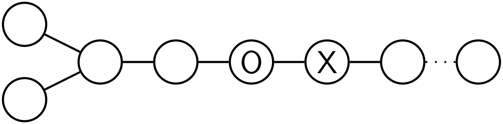
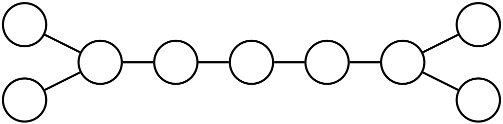


The Case with $P(2k+1,2)$

Result 6 Player One has a winning strategy on $P(2k+1,2)$.



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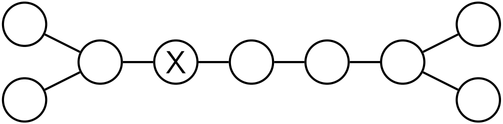

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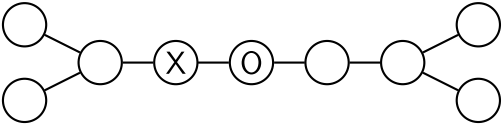

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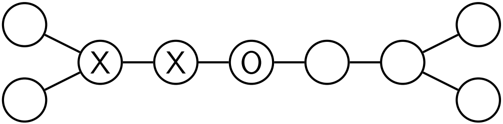

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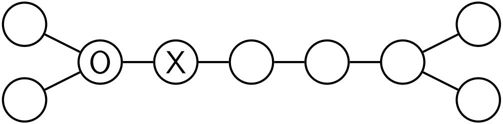

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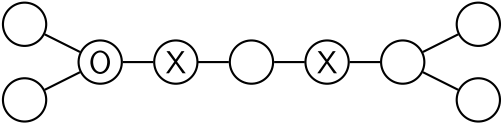

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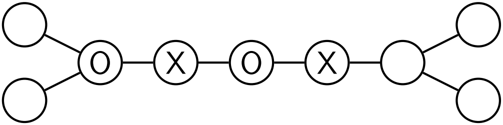

The Case with $P(2k+1,2)$

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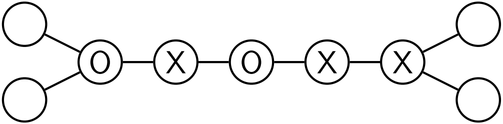
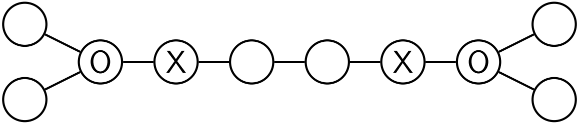


The Case with $P(2k, 2)$

Result 7 Both players have a drawing strategy on $P(2k, 2)$.

By the above argument, we can assume that Player Two takes both vertices of degree three and that Player One takes their neighbors of degree two.

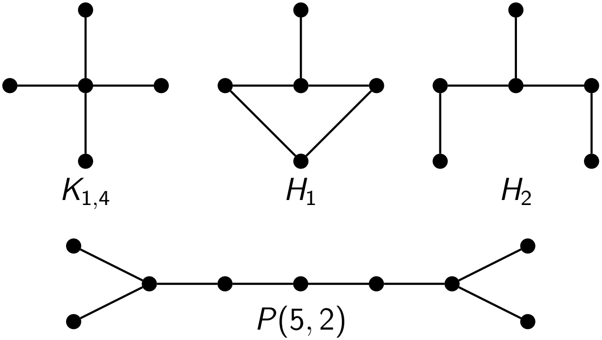


Since the path between the X's is of even length, eventually we reach a situation like in the figure above.

Our Main Result (Part 2)

Theorem

Player One has a winning strategy on G if and only if G has $K_{1,4}$, H_1, H_2, or $P(2k+1, 2)$ as a subgraph. Otherwise both players have a drawing strategy on G.



Two Variations

We also consider two variations.

In the *restricted* variation, only vertices that form a K_3 constitute a winning set.

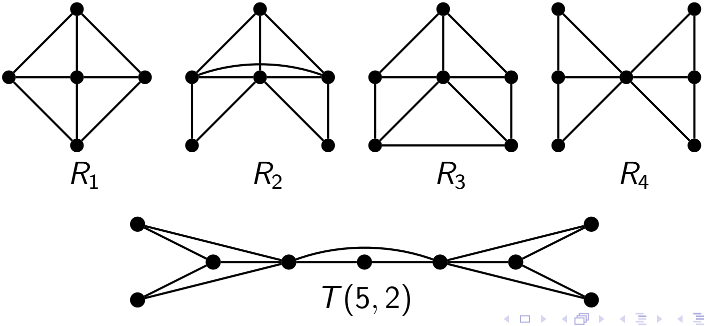
In the *induced* variation, only vertices that form an *induced* P_3 constitute a winning set.

In both cases we can provide a forbidden subgraph characterization.

Restricted Tic-Tac-Toe on Graphs

Theorem

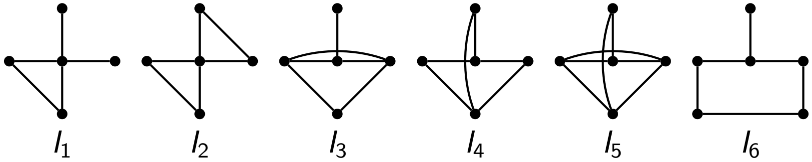
Player One has a winning strategy on G in restricted Tic-Tac-Toe if and only if G has R_1, R_2, R_3, R_4, or $T(2k+1,2)$ (arising from $P(2k+1,2)$) as a subgraph. Otherwise both players have a drawing strategy on G.



Induced Tic-Tac-Toe on Graphs

Theorem

Player One has a winning strategy on a graph G in induced Tic-Tac-Toe on graphs if and only if G contains one of the following as an induced subgraph: $K_{1,4}$, H_1, H_2, I_1, I_2, I_3, I_4, I_5, I_6, or any element of $\mathcal{I}(2k+1,2)$ (arising from $P(2k+1,2)$). Otherwise, both players have a drawing strategy on G.



Open Problems - The Pie Rule

The *pie rule* (also known as the swap rule or Nash's rule from Hex) is a common method for mitigating the advantage of going first [2]. If the pie rule is implemented, then after the first move is made, Player Two has one of two options:

(i) Player Two lets the move stand. Play then proceeds as normal.

(ii) Player Two "takes" that move. Player One then plays as if they were the second player.

What is the set of graphs in which each player has a winning strategy when the pie rule is implemented?

Open Problem - Double Move Version

As another way to neutralize Player One's advantage, we might also consider a variation in which *double moves* are allowed. There are several possibilities for variations in which double moves are allowed:

(i) On each turn, both players place two marks instead of one.

(ii) On Player One's first turn, they place a single mark. On Player Two's first turn, they place two marks. Play then proceeds as normal.

(iii) On Player One's first turn, they place a single mark. On each subsequent turn, both players place two marks.

(iv) Player Two places two marks for every mark placed by Player One.

For each of the above possibilities, determine necessary and sufficient conditions for each player to have a winning strategy on a graph G.

Open Problem - Full Play Convention

Suppose that we allow play to continue after one player has captured a P_3 (this is known as *full play convention* in [1, 3]).

(i) What is the set of graphs in which Player One cannot prevent Player Two from capturing a P_3?

(ii) What is the set of graphs in which Player One can prevent Player Two from capturing a P_3 only at the expense of capturing their own?

(iii) What is the set of graphs in which Player One can both capture a P_3 and prevent Player Two from capturing a P_3?

Open Problem - Misère Version

In a *misère* version of Tic-Tac-Toe, the first player to complete a (not necessarily induced) P_3 loses. What are the necessary and sufficient conditions for each player to have a winning strategy on the misère version of Tic-Tac-Toe on graphs?

Open Problem - Other Subgraphs

What if we were to consider other subgraphs (induced or otherwise) as our winning sets? In particular, if we were to generalize Connect-Four to graphs, then we would likely assume that both players were trying to capture a P_4. Likewise, we could generalize this further by assigning each player a family of graphs (which need not be the same for both players). The first player to capture any graph in their respective family wins.

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