Sensitivity Comparison of Two Seasonally Forced Epidemic Models

Jessica Lunsford and Dr. Ariel Cintron-Arias
Department of Mathematics and Statistics
East Tennessee State University

Abstract

Epidemiology is a tool for scientists, mathematicians, and others to use to access information and make predictions about the conditions that effect the outcome and behavior of disease within a population. Sensitivity analysis is an easily accessible method that can be used to help increase understanding, and to focus resources for research and decision making. With many types of epidemic models available, two models are considered. In the context of their purpose, scope, and relevance to the researcher, the information that can be extracted from numerical methods is described in relation to each compartmental epidemic model. The advantages, shortcomings, and relative quality of the traditional sensitivities are analyzed. Critical points, general trends, and the overall behavior are discussed as well as connections between mathematically described behavior and expected or observed behavior for each sensitivity to a parameter are studied such that conclusions can be made. By the definition of a limit, each sensitivity is used to determine the effect of a change in parameter values upon the rate of change in the system. However, the results vary depending upon the type of model used.

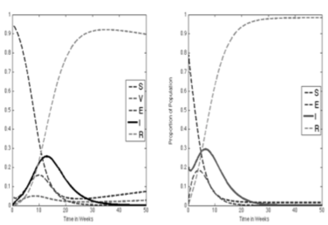
Introduction

Mathematical modeling in epidemiology is used to assist in decision making processes for both local and global healthcare providers. Once a model has been created, it can be improved upon and changed to represent different conditions. One of the goals of modeling in this manner is to create a model that is as dynamic as the disease itself, but that can also be used to predict and analyze different aspects of a disease within a population.

Sensitivity analysis is a way to look at the different parameters within a model, such that conclusions can be made about both a particular class of individuals as well as the parameters affecting the model. While there are many different forms of sensitivity analysis, the basic idea is that a change in a parameter is compared to a change in the system. Essentially, a sensitivity is showing the effect of a parameter under specified conditions. Each model behaves in a slightly different manner as we look at the perturbation of a parameter relative to a perturbation in a particular aspect of the nonlinear system.

The Models

The SVEIRS model (Figure 1) has five main compartments modeled by Equations 1-7. This system has state variables $f(t) = S(t), V(t), E(t), I(t), R(t)$ and parameters $\Theta = \mu, \xi, \beta_0, \phi, \theta, \kappa, \gamma, \sigma, \epsilon$ The population is a closed system, with the only method of exit being death and vaccination occurring at a steady rate. This model assumes a constant vaccination strategy. A table of parameter descriptions is found in Table 1. The SEIR model (Figure 2) has four compartments, modeled by Equations 8-12, This system has state variables $f(t) = S(t), E(t), I(t), R(t)$ and parameters $\Theta = \beta_0, \phi, \kappa, \gamma, \delta, \epsilon$ with the recovered class achieving permanent immunity. The model is subject to certain initial conditions such that about twenty percent of the population is immune, and are considered members of the recovered class. The epidemic is allowed to progress without new immunity by vaccination, thus adopting a herd immunity strategy. The numerical solutions to the PDE's are shown in (Figure 0). The solid line indicates the prevalence of infection.



$Fig.0$ The SEIR and SVEIRS Models Solutions Graphs

SVEIRS Model

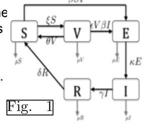


Fig. 1

$$(1)\quad \frac{dS}{dt} = \mu + \delta R - \beta SI - (\xi - \mu)S + \theta V$$
$$(2)\quad \frac{dV}{dt} = \xi S - \theta V - \epsilon\beta VI - \mu V$$
$$(3)\quad \frac{dE}{dt} = \beta SI + \epsilon\beta VI - (\kappa + \mu)E$$
$$(4)\quad \frac{dI}{dt} = \kappa E - (\gamma + \mu)I$$
$$(5)\quad \frac{dR}{dt} = \gamma I - (\delta + \mu)R$$
$$(6)\quad \beta(t) = \beta_0(1 + \epsilon\cos(\frac{2\pi t}{365} + \phi))$$
$$(7)\quad N = S + V + E + I + R$$

Table 1. Parameter Descriptions and Values for SVEIR Model

S_0	Proportion of population that is succeptible at t_0	.95
V_0	Proportion of population that is vaccinated at t_0	0
E_0	Proportion of population that is exposed to infection at t_0	.05
I_0	Proportion of population that are infected at t_0	0
R_0	Proportion of population that has recovered from disease at t_0	0
μ	death rate of individuals in all classes	$\frac{1}{365*75}$
ξ	rate of vaccination	.01
β_0	baselineinfection transmission rate	$\frac{2}{3}$
ϕ	changes the period of transmission	3
θ	vaccination failure rate	$\frac{1}{365}$
κ	rate at which exposed individuals become infective	.5
γ	recovery rate from infection	$\frac{1}{4}$
δ	rate at which recovered individuals lose immunity	$\frac{1}{365}$
ϵ	rate at which those vaccinated are capable of carrying the disease	$\frac{1}{2}$

SEIR Model

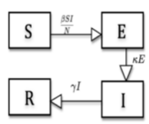


Fig. 2

$$(8)\quad \frac{dS}{dt} = -\beta(t)\frac{SI}{N}$$
$$(9)\quad \frac{dE}{dt} = \beta(t)\frac{SI}{N} - \kappa E$$
$$(10)\quad \frac{dI}{dt} = \kappa E - \gamma I$$
$$(11)\quad \frac{dR}{dt} = \gamma I$$
$$(12)\quad \beta(t) = (\beta_0(1 + \epsilon\cos(\frac{2}{365} + \phi))$$

Table 2. Parameter Descriptions for SEIR Model

N	Population total at t_0	10^7
S_0	Proportion of population that is susceptible at t_0	$n(1-V) - I_0$
E_0	Proportion of population that is exposed to infection at t_0	0
I_0	Proportion of population that are infected at t_0	1
R_0	Proportion of population that has recovered from disease at t_0	VN
V	Proportion of population that is vaccinated at t_0	.2
β_0	baselineinfection transmission rate	$\frac{2}{3}$
ϕ	changes the period of transmission	3
κ	rate at which exposed individuals become infective	.5
γ	recovery rate from infection	$\frac{1}{4}$
δ	rate at which recovered individuals lose immunity	$\frac{1}{365}$
ϵ	rate at which those vaccinated are capable of carrying the disease	$\frac{1}{2}$

Sensitivity Analysis

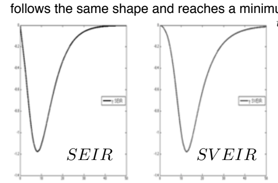
To create a traditional sensitivity, the partial differentials of the state variables are taken such that the change in each parameter is relative to the change of the state variable. This is possible by the definition of a limit as a rate of change. Because the solutions to the partial differential equations are the main driver behind the sensitivity analysis of both models. Each model equation is solved using numerical methods and computed in MATLAB. The partial derivatives are the Jacobian matrices of partial derivatives for the state variables and parameters

$$\frac{d}{dt}\frac{\partial x}{\partial \theta} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial g}{\partial \theta}$$

Here the focus is entirely upon the sensitivity of the infectious class in both models to shared or similar parameters

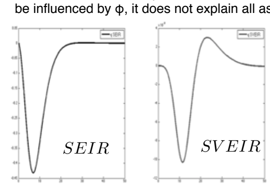
Sensitivity Comparison

The γ parameter shown in Figure 3 appears to be influenced largely by the prevalence of the infection. It is also notable that the initial behavior between the two models is slightly different. The SEIR model plummets in a linear fashion from t_0 to about $t = 8$. The SVEIR model is a little more concave, but follows the same shape and reaches a minimum at about $t = 12$. This is consistent with the observations of the prevalence of infection, and supports the idea of a strong link between the parameter γ and the Infectious class.



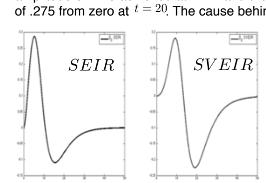
$Fig.3$ The SEIR and SVEIRS Models Sensitivity of the Prevalence of Infection to γ

The parameter ϕ shown in Figure 4 is generally used to adjust a model so that the peaks and valleys caused by seasonal forcing align with the data that has been acquired. The behavior of the two models with respect to this phase shift indicates that while the phase shift of the infectious class may be influenced by ϕ, it does not explain all aspects of the the SVEIRS model, but that is lacking in the SEIR model. It is likely that other parameters working in conjunction with with ϕ also serve to move the course of the infection



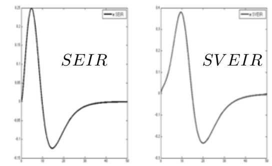
$Fig.4$ The SEIR and SVEIRS Models Sensitivity of the Prevalence of Infection to ϕ

The parameter β_0 shown in Figure 9 retains the same general shape in both models The steepest areas of ascent and descent for the models lie between t_0 and $t = 20$. As a cause of infection through transmission, this parameter appears to be the strongest influence in the SEIR model at the beginning, achieving a height of .3 at $t = 5$, with the effectiveness of disease transmission losing amplitude to a distance of around.1 from zero at $t = 15$. In the SVEIR model, the amplitude appears to behave in the opposite manner, with the lesser amplitude of .175 achieved at $t = 8$ and the greater amplitude of .275 from zero at $t = 20$. The cause behind the variation in amplitude can only be explained by influence from other classes or parameters. The traditional sensitivity methods used here may not be sufficient to explain this behavior, and other methods to support these conclusions are needed.



$Fig.5$ The SEIR and SVEIRS Models Sensitivity of the Prevalence of Infection to β_0

An analysis of κ supports the conclusion that the herd immunity strategy found in the SEIR model causes a sharper increase in the prevalence of infection, but shows that the period of outbreak is longer for the SVEIR model. This is shown where the SEIR model in Figure 6 where at $t = 30$ the SEIR model appears to be leveling out to zero. In contrast, the SVEIR model at $t = 30$ is still rising to zero. This raises questions about time dependence between exposed individuals and the infective class. It would be interesting to analyze the proportion of those considered immune to the disease as compared to those simply exposed.



It may be a good place to analyze whether or not prevention of disease transmission would be a viable strategy over transmission.

$Fig.6$ The SEIR and SVEIRS Models Sensitivity of the Prevalence of Infection to κ

Discussion

These two models share some of the same characteristics, and yet as a result of their parameters, are much different. The SVEIRS model and the SEIR model follow similar paths in their solutions however their strategies for vaccination of the infectious class are not the same. The SVEIRS model works in a way similar to to administering flu vaccines in the United States. Over time, a portion of the population is vaccinated, but in time, the vaccination is not effective forever, so persons may become infected or exposed to the disease in spite of vaccination. The SEIR model is attempting to work on the premise of what is called herd immunity. This is similar to the vaccination of young children here in the United States. The premise behind herd immunity is that if a large enough proportion of the population is vaccinated before a disease arrives, then it will prevent widespread prevalence of the disease. Several observations can be made about these two strategies from the perspective of their solutions. Most of the Infectious activity is occurring early in the cycle, and indicates that the disease is only affecting about thirty percent of the population.

Conclusion

The SVEIRS model's prevalence of infection lasts longer and is more gradual, where the SEIR model subsided much faster and begins more rapidly. The sudden onset of an epidemic can strangle an economy, or even have disastrous effects upon the healthcare system. However their seems to be a tradeoff in that almost all of the SEIR population does attain permanent immunity. In the SVEIR model, The recovered class becomes susceptible, and it is likely that the infection will return

Future Work

A fitting of the models to existing data would be recommended so that vaccination strategy may pinpoint the most effective parameters to address. A cost-benefit analysis would be a useful tool to have in reference to discern which strategy to use on a given population. So too would an analysis of the existing infrastructure's capability of implementing either strategy. Applications within game theory may also shed some light on the subject of whether the models' vaccination strategies would be realistic, and a study of the network structures of both the population and healthcare systems would help to refine the model to a more realistic and viable model.

Acknowledgements

Many thanks to the Research Discovery Position in Mathematics and Statistics funded by the East Tennessee State University Honors College Research Discovery Program.

References

[1] Bai et al...., Stochastic and Deterministic Models for Agricultural Production Networks. Mathematical Biosciences. 373-402, Volume 4, Number 3, July 2007.
[2] F. Brauer. Mathematical Models in Population Biology and Epidemiology. Springer-Verlag New York. 2001.
[3] A. Capaldi et al. Parameter Estimation and Uncertainty Quantification for an Epidemic Model
[4] Z. Feng et al.... Modeling the Effects of Vaccination and Treatment on Pandemic Influenza. The AAPS Journal, May 2011.
[5] Grassly et al., Seasonal Infectious Disease Epidemiology. Proceedings of the Royal Society.273:2541-2550,7 July 2006.SENSITIVITY COMPARISON OF TWO SEASONALLY FORCED EPIDEMIC MODELS19
[6] K. Tomaseth et al, Generalized Sensitivity Functions in Physiological System Identification. Annals of Biomedical Engineering, Vol. 27, pp. 607616, 1999.