

Chapter 10. Introducing Probability

The Idea of Probability

Note. The text states: “Our purpose in this chapter is to understand the language of probability, but without going into the mathematics of probability theory. ... The big fact that emerges in this: **chance behavior is unpredictable in the short run but has a regular and predictable pattern in the long run.**” (page 247) In other words, “the probability of an outcome of a random phenomenon is the proportion of times that outcome would occur in a very long series of repetitions.” (page 263)

Example. Example 10.2 page 247. Suppose a coin is tossed. Then the probability the coin comes out heads (H) is $1/2 = 0.5$ and the probability that it comes out tails (T) is $1/2 = 0.5$. Since probability is an expected proportion which results from repeating an experiment many times (theoretically, an *infinite* number of times), then we would expect that if we tossed a coin many, many times, then half of the tosses should come out heads and the other half should come out tails. The following figure indicates the outcomes of two trials of performing this

experiment 5000 times. Notice that the curves are not very close to the expected $1/2 = 0.5$ value when the number of coin tosses is small (less than about 25 tosses), but as the number of tosses gets large, both curves get very close to $1/2 = 0.5$.

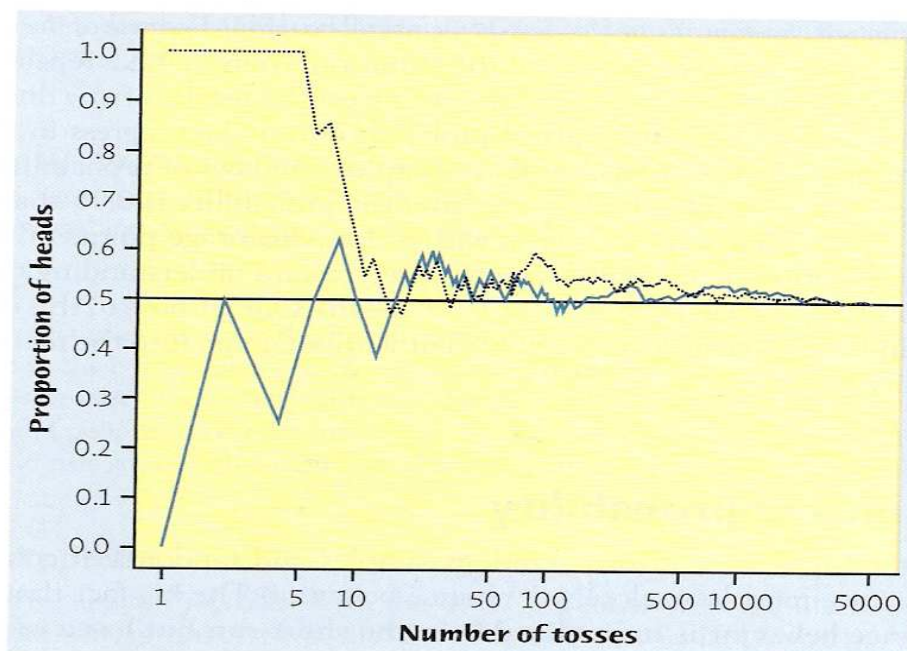


Figure 10.1 Page 248

Definition. We call a phenomenon **random** if individual outcomes are uncertain but there is nonetheless a regular distribution of outcomes in a large number of repetitions. The **probability** of any outcome of a random phenomenon is the proportion of times the outcome would occur in a very long series of repetitions.

Probability Models

Note. To describe a probability model for a phenomenon (or an “experiment”), we will need two things: a list of all possible outcomes of the experiment, and a probability of each outcome.

Definition. The **sample space** S of a random phenomenon is the set of all possible outcomes. An **event** is an outcome or a set of outcomes of a random phenomenon. That is, an event is a subset of the sample space. A **probability model** is a mathematical description of a random phenomenon consisting of two parts: a sample space S and a way of assigning probabilities to events.

Example. Example 10.4 Page 251. Suppose we roll two six sided dice, one die red and the other die green. Then there are 36 possible outcomes:

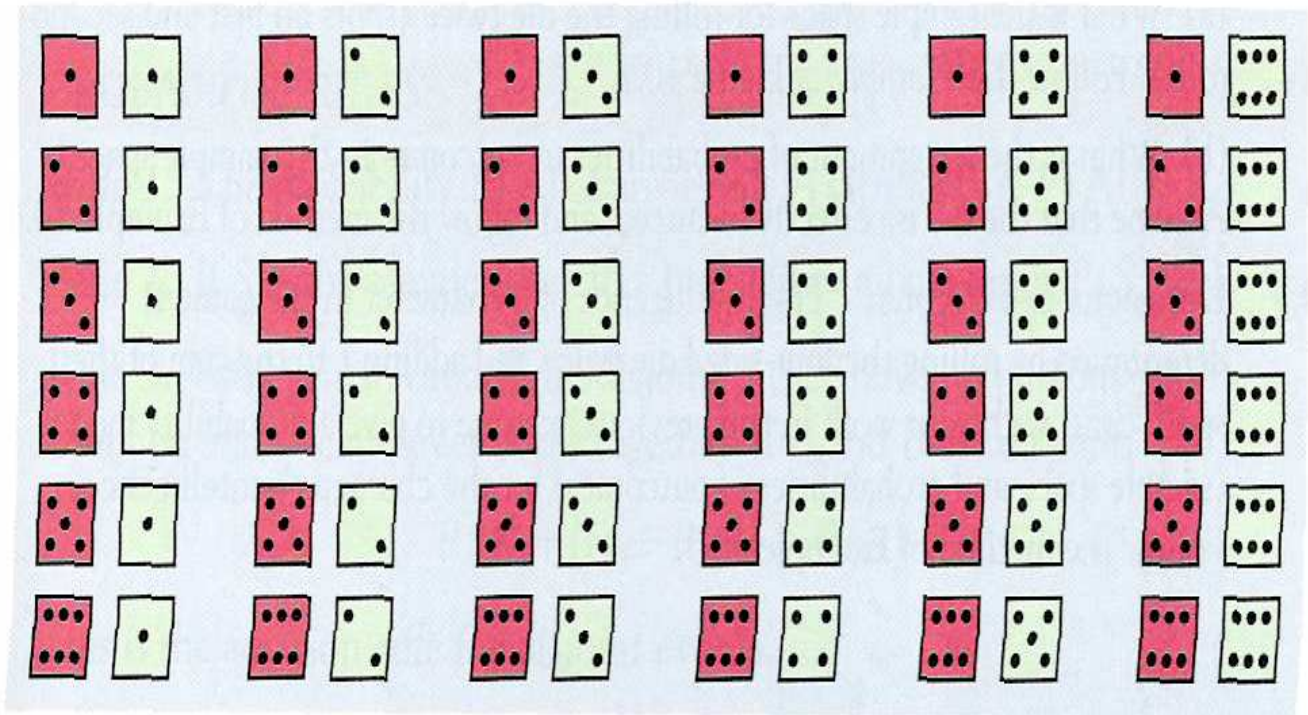


Figure 10.2 Page 251

The sample space is the 36 outcomes in the figure. Each outcome is equally likely, so the probability of each of these events is $1/36$.

Example. Example 10.5 page 252. In the experiment from the previous example, we are interested in the numerical outcome of each roll of the dice. In this case, the sample space is $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Since there are 36 equally likely outcomes from the previous example, we can count the number of outcomes which yield each element of the sample space to find probabilities. For example, there are two different ways to roll a 3 and so the probability of rolling a 3 is $P(3) = 2/36$. There are three ways to roll a 4, so $P(4) = 3/36$. The probability model for this experiment is:

Number of spots	2	3	4	5	6	7	8	9	10	11	12
Probability	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	3/36	1/36

Notice that the most likely outcome is rolling a 7.

Probability Rules

Note. Some facts that must be true for *any* assignment of probabilities include:

1. Any probability is a number between 0 and 1.
2. All possible outcomes together must have probability 1.
3. If two events have no outcomes in common, the probability that one or the other occurs is the sum of their individual probabilities.
4. The probability that an event does not occur is 1 minus the probability that the event does occur.

Note. On page 253, the text states that: “An event with probability 0 never occurs, and an event with probability 1 occurs on every trial.” **This is not true!** It will be true with the examples we see in here, but it is not always true. Consider the experiment in which a coin is tossed an infinite number of times (the infinite part is necessary to illustrate this property). The probability of the coin coming out heads every time is 0, but this outcome is possible. In fact, the probability of *any* particular outcome is 0, so whatever the outcome is,

it has probability 0. To reiterate, **probability 0 does not mean impossible** and probability 1 does not mean certain.

Note. We often represent an event by a capital letter from near the beginning of the alphabet. If A is an event, we write the probability of the event as $P(A)$. With this notation, the above four rules can be written as the following probability rules:

Rule 1. The probability $P(A)$ of any event A satisfies $0 \leq P(A) \leq 1$.

Rule 2. If S is the sample space in a probability model, then $P(S) = 1$.

Rule 3. Two events A and B are **disjoint** if they have no outcomes in common and so can never occur together. If A and B are disjoint,

$$P(A \text{ or } B) = P(A) + P(B).$$

This is the **addition rule for disjoint events**.

Rule 4. For any event A ,

$$P(A \text{ does not occur}) = 1 - P(A).$$

Example S.10.1. Probably Curly.

A survey is given to a population of 100 Stooge fans which asks them which is their favorite “third stooge,” Curly (C), Shemp (S), Joe (J), or Curly Joe (CJ). 60 of them choose Curly, 25 of them choose Shemp, 10 of them choose Joe, and 5 of them choose Curly Joe.

- (a) Based on the survey, for this population what are: $P(C)$, $P(S)$, $P(J)$, and $P(CJ)$?
- (b) What is the probability that a member of this population chooses Curly or Shemp as their favorite Stooge?
- (c) What is the probability that a member of this population does not choose Curly as their favorite Stooge?

Discrete Probability Models

Definition. A probability model with a finite sample space is called **discrete**.

Note. It is also possible for an infinite sample space to be discrete. For example, suppose you flip a coin until it comes up heads and then stop. The number of times the coin is flipped is discrete and the sample space is $S = \{1, 2, 3, 4, \dots\} = \mathbb{N}$. The probability model is

Number of tosses	1	2	3	4	$\dots$	n	$\dots$
Probability	$1/2$	$1/4$	$1/8$	$1/16$	$\dots$	$1/2^n$	$\dots$

If you have had experience with summing series (or with Zeno's Paradox), then you might know that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots = 1.$$

So this really is a probability model.

Example. Exercise 10.11 page 256. See Example 10.7 for the basic probabilities for Benford's Law.

Continuous Probability Models

Definition. A **continuous probability model** assigns probabilities as areas under a density curve. The area under the curve and above any range of values is the probability of an outcome in that range.

Note. The text states on page 258 that “all continuous probability models assign probability 0 to every individual outcome.” You might think that this contradicts the statement from page 253 that “all possible outcomes together must have probability 1.” However, in a continuous probability model, it is not *summation* but *integration* (a concept from calculus) which applies to the calculation of probabilities. For more details, take “Probability and Statistics, Calculus Based” (MATH 2050).

Example. Exercise 10.13 page 259. This exercise uses the uniform density curve $f(x) = 1$ for $0 \leq x \leq 1$.

Random Variables

Definition. A **random variable** is a variable whose value is a numerical outcome of a random phenomenon. The **probability distribution** of a random variable X tells us what values X can take and how to assign probabilities to those values.

Example S.10.2. Continuously Distributed Stooges.

Recall from Example S.3.2 that we assumed the number of slaps per film in the Three Stooges film to be normally distributed with a mean of $\mu = 12.95$ and standard deviation $\sigma = 4.50$ (that is, the distribution is $N(12.95, 4.50)$). Denote the count of slaps per film by the letter F . Then F is a random variable and its probability distribution is $N(12.95, 4.50)$.

- (a) Under this assumption, if a Stooges film is chosen at random, what is the probability that the film has a number of slaps between 15 and 20? That is, what is $14 \leq P(F) \leq 20$?
- (b) What is the probability that a film chosen at random has more than 14 slaps per film? That is, what is $P(F) \geq 14$?

Note. Gamblers often use odds to describe outcomes. If “the odds of event A is a to b ” ($a : b$), then the probability of event A is $P(A) = b/(a + b)$. For example, if the odds for event A are 3 to 1, the probability of A is $P(A) = 1/(3 + 1) = 1/4$. Whereas probability must be between 0 and 1, the odds of an event are between 0 and infinity.

Personal Probability

Definition. A **personal probability** of an outcome is a number between 0 and 1 that expresses an individual’s judgment of how likely the outcome is.

Note. We don’t explore personal probability any further.

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