Chapter 4. Polynomial and Rational Functions

4.6. The Real Zeros of a Polynomial Function

Note. In preparation for this section, you may need to review Sections R.1, R.5, R.6, and Section 1.2.

Note. In this section, we find zeros of polynomials analytically and approximate them numerically. This will allow us to factor polynomials with the ultimate goal of stating the Fundamental Theorem of Algebra (in the final section).

Theorem. Division Algorithm for Polynomials. If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is not the zero polynomial, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \text{ or } f(x) = q(x)g(x) + r(x)$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$. $f(x)$ is the *dividend*, $g(x)$ is the *divisor*, $q(x)$ is the *quotient*, and $r(x)$ is the *remainder*.

Note. We can use the Division Algorithm to show:

Theorem. Remainder Theorem. Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Theorem. Factor Theorem. Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

Note. The Factor Theorem tells us that *finding zeros of a polynomial* and *factoring a polynomial* are **exactly the same problem**!

Example. Page 375 numbers 12 and 14.

Note. Next, we are interested in counting the zeros of a polynomial. First we have:

Theorem. Number of Real Zeros. A polynomial function cannot have more real zeros than its degree.

More specifically:

Theorem. Descartes' Rule of Signs.

Let f denote a polynomial function written in standard form.

The number of positive zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(x)$ or else equals that number less an even integer.

The number of negative zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(-x)$ or else equals that number less an even integer.

Example. Page 375 numbers 22 and 30.

Note. Next, we have a result which gives possible rational zeros of *certain* polynomials (those with integer coefficients):

Theorem. Rational Zeros Theorem. Let f be a polynomial function of degree 1 or higher of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0, \quad a_0 \neq 0$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0, and q must be a factor of a_n.

Example. Page 375 numbers 34 and 40.

Note. Once we find a zero of a polynomial, we can then use the Factor Theorem to factor it and get a smaller degree polynomial. We can then use the relevant part of the factored polynomial (which when set equal to zero is called the *depressed equation*) to find additional zeros.

Example. Page 375 number 46.

Note. In summary, the book states the Steps for Finding the Real Zeros of a Polynomial Function:

Step 1: Use the degree of the polynomial to determine the maximum number of zeros.

Step 2: Use Descartes' Rule of signs to determine the possible number of positive zeros and negative zeros.

Step 3: **(a)** If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros.

(b) Use substitution or long division to test each potential rational zero.

(c) Each time that a zero (and thus a factor) is found, repeat step 3 on the depressed equation.

Step 4: In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

Example. Page 375 number 62, page 376 number 74.

Definition. A second degree polynomial with real coefficients ax^2+bx+c is an *irreducible quadratic* if it cannot be factored over the real numbers.

Note. An irreducible quadratic has no real zeros. An example is x^2+9. These types of polynomials are important in factoring polynomials:

Theorem. Every polynomial function with real coefficients can be uniquely factored into a product of linear factors and/or irreducible quadratic factors.

Corollary. A polynomial function with real coefficients of odd degree has at least one real zero.

Definition. A positive number M is a *bound on the zeros* of a polynomial if any zero r of the polynomial satisfies $|r| \leq M$. This means that all the real zeros satisfy: $-M \leq$ any real zero of $f \ \leq M$.

Theorem. Bounds on Zeros. Let f denote a polynomial function whose leading coefficient is 1:

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

A bound M on the zeros of f is the smaller of the two numbers

$\text{Max}\{1, |a_0|+|a_1|+|a_2|+\cdots+|a_{n-1}|\}$ and $1+\text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}$

where Max{ } means "choose the largest entry in { }."

(This is also valid for the complex zeros of f.)

Example. Page 376 number 84.

Note. The following (which is valid for any continuous function) is useful in approximating zeros of polynomials.

Theorem. Intermediate Value Theorem. Let f denote a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, then there is at least one zero of f between a and b.

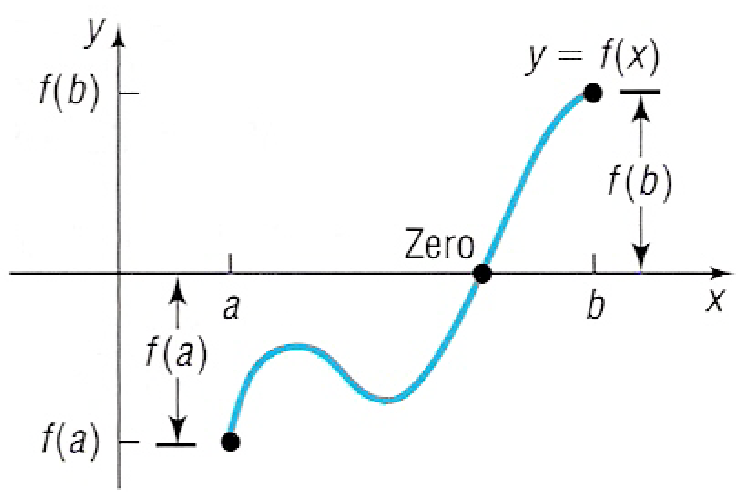

Figure 59 Page 372.

Note. We can use the Intermediate Value Theorem to approximate zeros as follows:

Step 1: Find two consecutive integers a and $a+1$ such that f has a zero between them.

Step 2: Divide the interval $[a, a+1]$ into 10 equal subintervals.

Step 3: Evaluate f at each endpoint of the subintervals until the Intermediate Value Theorem applies; this interval then contains a zero.

Step 4: Repeat the process starting at Step 2 until the desired accuracy is achieved.

Example. Page 373 Example 10, page 376 number 104.