Chapter R. Review

R.5. Factoring Polynomials

Note. It is assumed that you know the properties of addition and multiplication as explained in Section R.1. If you are not comfortable with this, then please review (especially pages 10–13).

Note. We begin our adventure with a few definitions!

Definition. The *integers* are

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The *rational numbers* (or "fractions") are

$$\mathbb{Q} = \left\{ \frac{p}{q} \text{ such that } p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$

definition. A *polynomial* in one variable x is an algebraic expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are constants called *coefficients* of the polynomial, $n \geq 0$ is an integer (i.e., n is some element of the set $\{0, 1, 2, \ldots\}$), and x is a variable. If $a_n \neq 0$, then a_n is the *leading coefficient* and n is the *degree* of the polynomial.

Example. Some examples of polynomials are:

$p(x)$	degree n
4	0
7	0
$4x+2$	1
$2x+4$	1
$7x^2-4x+\pi$	2
$97x^{41}-24x^2$	41
$9,475,025x^3-0.7814$	3

Example. Some functions which are **not** polynomials are:

$$\begin{aligned} f_1(x) &= \sqrt{x} = x^{1/2} \\ f_2(x) &= \sqrt[3]{x} = x^{1/3} \\ f_3(x) &= x^\pi \\ f_4(x) &= 1/x \\ f_5(x) &= 7x^3+2x^2-5x+4+x^{1.1} \end{aligned}$$

Recall. A *monomial* is a polynomial with all but one coefficient 0: ax^k for $a \neq 0$ and $k \geq 0$. A *binomial* is a polynomial with two nonzero coefficients: $ax^k + bx^m$ where $a \neq 0$, $b \neq 0$, $k > 0 > m \geq 0$, and $k \neq m$ are integers. A *trinomial* is a polynomial with exactly three nonzero coefficients: $ax^k + bx^m + cx^n$ where $a \neq 0$, $b \neq 0$, $c \neq 0$, $k > m > n \geq 0$, and $k \neq m \neq n \neq k$ are integers.

Note. We are mostly concerned with binomials of the form $mx+b$. these are also called polynomials of degree 1. Our goals are to find when such equations are 0 (Section 1.1) and to graph such equations (Section 4.1).

Note. We are also interested in trinomials of the form ax^2+bx+c. This is also called a *polynomial of degree* 2. Our goals are to find out when such a polynomial is 0 (Section 1.2), to graph such polynomials (Sections 3.4 and 3.5), and applications of such polynomials (Section 4.1).

Definition. A binomial of degree one, $ax+b$, is a *factor* of a polynomial $p(x)$ if $p(x) = (ax+b)r(x)$ for a polynomial $r(x)$. If a, b, and the coefficients of $p(x)$ and $r(x)$ are all integers, then $ax+b$ is a *factor over the integers* of $p(x)$.

Example. Since $(5x-2)(2x+3) = 10x^2+11x-6$ (Right? Remember FOIL!), then $5x-2$ and $2x+3$ are factors of $10x^2+11x-6$.

Definition. If a polynomial cannot be written as the product of two other polynomials (excluding 1 and -1), then the polynomial is said to be *prime*. When a polynomial is written as a product consisting only of prime factors, it is said to be *factored completely*.

Example. We can completely factor $30x^2 + 33x - 18$ as:

$$30x^2 + 33x - 18 = 3(10x^2 + 11x - 6) = 3(5x - 2)(2x + 3).$$

Notice that we hae used information we already knew (given in the example above), we have used equal signs when things are equal, and we have not written any unnecessary or incorrect symbols! That is, we have clearly and cleanly communicated our computations!!! **Mathematics consists of these two very important properties: accuracy and clarity!!!**

Example. Page 51 number 6.

Note. Based on FOIL, we have the following algebraic *identities* (i.e., equations which hold for every value of the variable x and for any values of the coefficient a):

$$\begin{aligned} x^2 - a^2 &= (x - a)(x + a) \\ x^2 + 2a + a^2 &= (x + a)^2 \\ x^2 - 2ax + a^2 &= (x - a)^2 \\ x^3 + a^3 &= (x + a)(x^2 - ax + a^2) \\ x^3 - a^3 &= (x - a)(x^2 + ax + a^2) \end{aligned}$$

Example. Page 51 numbers 20, 26, 34 ,and 36.

Note. We will see in the not-to-distant future that we can factor any second degree polynomial $ax^2 + bx + c$. For now, we concentrate on second degree polynomials of the form $x^2 + Bx + C$.

Note. To factor $x^2 + Bx + C$, where B and C are integers, find integers a and b whose product is C and whose sum is B. That is, find integers a and b where $B = a + b$ and $C = ab$. Then $x^2 + Bx + C = (x + a)(x + b)$.

Example. Page 46 Example 9. Page 51 numbers 40 and 48.

Note. Until the very last section of this course (Section 4.7), we will only study real numbers $\mathbb{R}$ (as opposed to complex numbers $\mathbb{C}$). Whenever a real number a is squared, the result is nonnegative: $a^2 \geq 0$ for all $a \in \mathbb{R}$. Therefore we have:

Theorem. Any polynomial of the form $x^2 + a^2$, a real, is prime.

Note. One can show that every polynomial with real coefficients can be factored into prime factors which consist of prime first degree polynomials and prime second degree polynomials with real coefficients (Theorem on page 37, Section 4.6).

Note. We can also factor polynomials "by grouping." In this method we recognize common factors and take advantage of the distributive law of multiplication over division.

Example. Page 51 numbers 52 and 56.

Note. Now we tackle the task of factoring the second degree polynomial $Ax^2 + Bx + C$ where $A \neq 1$. We follow these steps:

1. Find the value of AC.

2. Find integers with product AC that add up to B. That is, find integers ab such that $AC = ab$ and $B = a + b$.

3. Write $Ax^2 + Bx + C = Ax^2 + ax + bx + C$.

4. Factor by grouping.

Example. Page 51 numbers 62, 68, 100, and 124.