Chapter 2. Functions and Their Graphs

2.1. Functions

Note. In preparation for this section, you may need to review Appendix A, Sections A.1 and A.8.

Definition. A *relation* between two sets X and Y is a set of ordered pairs of elements of the two sets. If $x \in X$, $y \in Y$, and (x, y) is in the relation, then we say x *corresponds* to y (or that y *depends on* x and sometimes write $x \to y$).

Example. We could define a relation between the people in this class and the makes of cars we drive. One element of this relation would be (Dr. Bob, Volkswagen). Another relation is probably (x, Volkswagen) where x is one of you. If so, then we can say "Dr. Bob corresponds to Volkswagen" and "x corresponds to Volkswagen," or Dr. Bob $\to$ Volkswagen and $x \to$ Volkswagen.

Notice in a relation that more than one element of X (in this example, Dr. Bob and x) may correspond to the same element of Y (in this case, Volkswagen). I also reluctantly drive a Plymouth van. So another element of the relation is (Dr. Bob, Plymouth), or Dr. Bob $\to$ Plymouth. So, in a relation, one element of X (in this example, Dr. Bob) may correspond to more than one element of Y (in this example, Volkswagen and Plymouth).

Additional observations are that some elements of X may not correspond to anything in Y (some of you may not drive, say). Also, some elements of Y may not depend on any element of X (none of us may drive a Mercedes, say).

Definition. Let X and Y be two nonempty sets. A *function* from X to Y is a relation that associates with each element of X exactly one element of Y. The set X is the *domain* of the function. If $x \in X$, $y \in Y$, and x corresponds to y in the function, then y is the *value* of x. The *range* of the function is the set of all elements of Y which depend on elements of X.

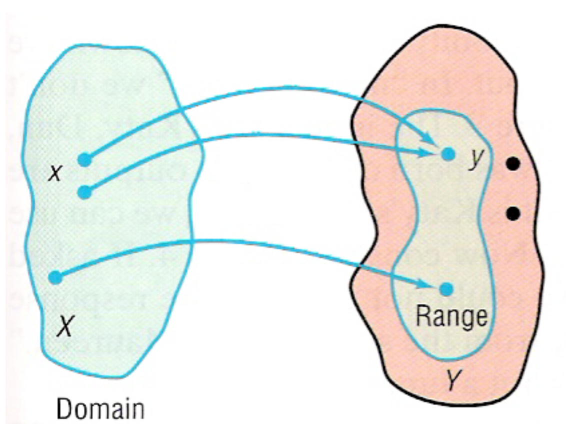

Figure 2, Page 51

Note. Our example about cars is a relation, but not a function. The element of X called "Dr. Bob" corresponds to more than one element of Y.

Note. The text says: "The idea behind a function is its predictability. If the input is known, we can use the function to determine the output. In 'nonfunctions' we don't have this predictability." Or briefly: "For a function, no input has more than one output."

Note. It's common to think of a function as a machine. You give it a good input (that is, an element of the domain) and it gives you a single output (that is, element of the range).

Example. The square root key on a scientific calculator represents a function. You give it a number and it outputs a (single) value. Since we cannot take square roots of negatives in here, the domain of the function is all nonnegative real numbers:

$$\{x \mid x \geq 0\} = [0, \infty).$$

What's the range?

Examples. Page 61 numbers 18 and 22.

Note. We often denote a function with a letter. If f is a function from set X to set Y and x corresponds to y, we write $y = f(x)$, read "y equals f of x." Again, the machine concept applies and we have input x, machine f, and output y (or $f(x)$):

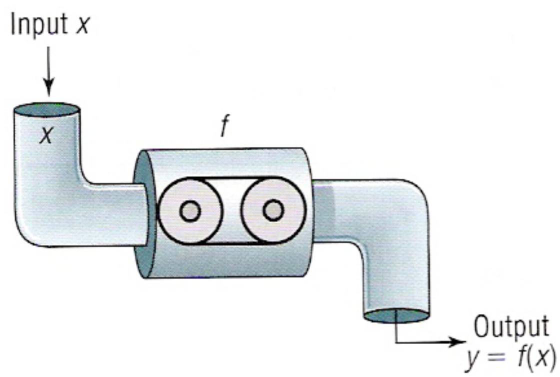

Figure 6, Page 54

Definition. We abbreviate the statement f is a function from set X into set Y as $y = f(x)$ where x is a variable element of set X (called the *independent variable*) and y is a variable element of set Y (called the *dependent variable* because its value depends on x).

Examples. Page 61 number 28b,c,g,h, and number 30a,h.

Definition. When a function f is defined by an equation in x and y, we say that the function f is given *implicitly* by the equation.

Note. To see if y is a function of x, all we have to do is solve the equation for y and see if we get an expression which will represent a single y value for each "good" input value x.

Note. Not all equations in x and y define a function. See Example 7 on page 56. WARNING: The ideas being discussed here are different from the idea of a function being "implicit to an equation" as encountered in Calculus 1. See http://www.etsu.edu/math/gardner/1110/c2s6.pdf for details.

Note. To find the domain of a function, we need to determine which x-values are "good" values. The easiest way to do this is to find which x values are "bad," and throw them out. At this stage, the only algebraic manipulations we cannot perform are:
(1) division by 0, and
(2) square roots of negative.
There will be other concerns later, such as logarithms of negatives and inverse sines of numbers greater than 1, but for now these two problems are our only constraints.

Example. Page 62 number 58.

Definition. We can perform the following operations on two functions f and g:

(1) The *sum* $f + g$ is $(f + g)(x) = f(x) + g(x)$.

(2) The *difference* $f - g$ is $(f - g)(x) = f(x) - g(x)$.

(3) The *product* $f \cdot g$ is $(f \cdot g)(x) = f(x) \cdot g(x)$.

(4) The *quotient* $\frac{f}{g}$ is $\left(\dfrac{f}{g}\right) = \dfrac{f(x)}{g(x)}$, when $g(x) \neq 0$.

Examples. Page 62 numbers 66 and 76.

Example. Page 62 number 90.