Chapter 2. Limits and Continuity

2.4 One-Sided Limits

Definition. Informal Definition of Right-Hand and Left-Hand Limits.

Let $f(x)$ be defined on an interval (a, b), where $a < b$. If $f(x)$ approaches arbitrarily close to L as x approaches a from within that interval, then we say that f has *right-hand limit* L at a, and write

$$\lim_{x \to a^+} f(x) = L.$$

Let $f(x)$ be defined on an interval (c, a), where $c < a$. If $f(x)$ approaches arbitrarily close to M as x approaches a from within the interval (c, a), then we say that f has *left-hand limit* M at a, and we write

$$\lim_{x \to a^-} f(x) = M.$$

Definition. Formal Definitions of One Sided Limits.

We say that $f(x)$ has *right-hand limit* L *at* x_0, and write

$$\lim_{x\to x_0^+} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that $f(x)$ has *left-hand limit* L *at* x_0, and write

$$\lim_{x\to x_0^-} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Example. (Example 3 page 87) Prove that $\lim_{x\to 0^+} \sqrt{x} = 0$.

Proof. Let $\epsilon > 0$ be given. We have $x_0 = 0$ and $L = 0$. Choose $\delta = \epsilon^2$. Then if $0 < x - x_0 = x - 0 = x < \delta = \epsilon^2$, we have $\sqrt{x} < \epsilon$, or $|\sqrt{x} - 0| < \epsilon$, or $|f(x) - L| < \epsilon$. Therefore, the result holds. *QED*

Example. Consider limits as x approaches -1 and $+1$ for $f(x) = \sqrt{1-x^2}$.

Theorem 6. Relation Between One-Sided and Two-Sided Limits

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Example. Page 91 number 10.

Example. Page 92 number 46.

Theorem 7.

For θ in radians,

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

Proof. (Informally, notice the graph of $(\sin \theta)/\theta$ on page 88—the result is no surprise when you consider Dr. Bob's Anthropomorphic Definition of Limit.) Suppose first that θ is positive and less than $\pi/2$. Consider the picture:

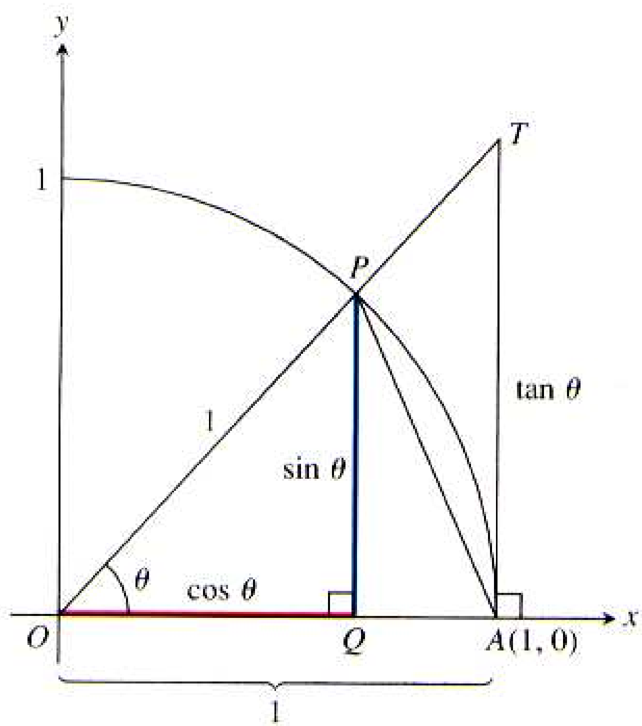

Figure 2.33, page 89

Notice that

$$\text{Area } \Delta OAP < \text{area sector } OAP < \text{area } \Delta OAT.$$

We can express these areas in terms of θ as follows:

$$\text{Area } \Delta OAP = \tfrac{1}{2} \text{ base} \times \text{height} = \tfrac{1}{2}(1)(\sin\theta) = \tfrac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(1)^2\theta = \tfrac{\theta}{2}$$

$$\text{Area } \Delta OAT = \tfrac{1}{2} \text{ base} \times \text{height} = \tfrac{1}{2}(1)(\tan\theta) = \tfrac{1}{2}\tan\theta.$$

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the positive number $(1/2)\sin\theta$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Since $\lim_{\theta\to 0^+} \cos\theta = 1$, the Sandwich Theorem gives

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

Since $\sin\theta$ and θ are both odd functions, $f(\theta) = \dfrac{\sin\theta}{\theta}$ is an even function and hence $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin\theta}{\theta}$. Therefore

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta},$$

so $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ by Theorem 4. *QED*

Example. Page 89 example 5a: Show that $\displaystyle\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$.

Solution. We use the trig identity $\cos h = 1 - 2\sin^2(h/2)$ (a half-angle identity). First, we have

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h}.$$

Now, replacing $h/2$ with θ we get

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h} = -\lim_{\theta\to 0} \frac{\sin\theta}{\theta} \sin\theta = -(1)(0) = 0.$$

QED

Example. Page 92 number 26.