Chapter 3. Differentiation

3.2 The Derivative as a Function

Definition. Derivative Function.

The *derivative* of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Note. Motivated by Sections 2.1 and 3.1, we see that $f'(x)$ is the slope of the line tangent to $y = f(x)$ as a function of x.

Note. There are a number of ways to denote the derivative of $y = f(x)$:

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}[f].$$

Example. Page 128 Example 2(a). Notice the text uses the "alternative formula" of the derivative.

Examples. Page 132 numbers 14 and 30.

Note. We can also study "one-sided derivatives" at a point defined as follows:

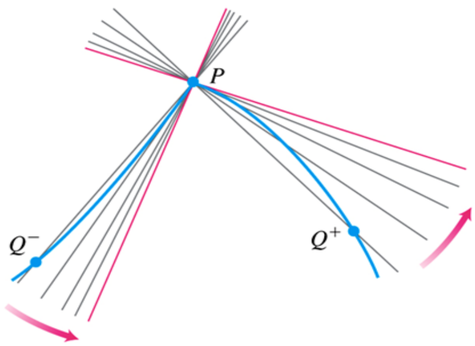
$$\text{Right-hand derivative at } a: \quad \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

$$\text{Left-hand derivative at } b: \quad \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$$

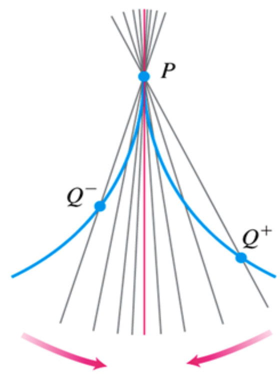
Example. Page 133 number 40.

Note. The function in the previous example is not differentiable at $x = 1$.

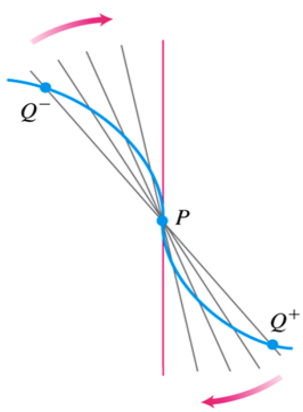
There are a number of reasons as to why a function might not have a derivative at a point. Some of these reasons are illustrated here:



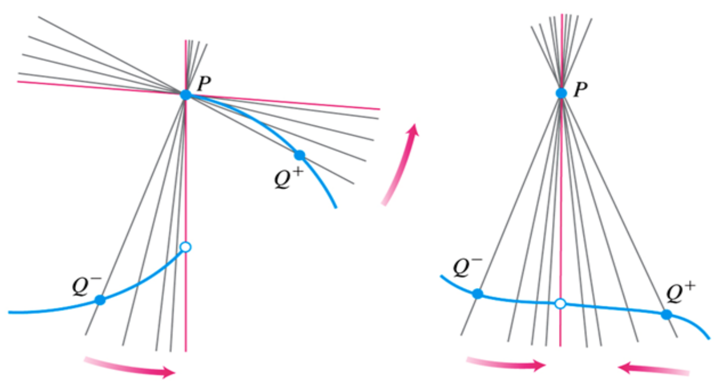
1. a *corner*, where the one-sided derivatives differ.



2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.



3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



4. a *discontinuity* (two examples shown).

From page 130

Theorem 1. Differentiability Implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof. By definition, we need to show that $\lim_{x \to c} f(x) = f(c)$, or equivalently that $\lim_{h \to 0} f(c+h) = f(c)$. Then

$$\begin{aligned}
\lim_{h \to 0} f(c+h) &= \lim_{h \to 0} \left(f(c) + \frac{f(c+h) - f(c)}{h} \cdot h \right) \\
&= \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h \to 0} h \\
&= f(c) + f'(c) \cdot 0 \\
&= f(c).
\end{aligned}$$

Therefore f is continuous at $x = c$. *QED*

Examples. Page 155 numbers 48 and 54.