Chapter 3. Differentiation

3.3 Differentiation Rules

Derivative of a Constant Function.

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}[c] = 0.$$

Proof. From the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to o} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

QED

Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Note. Before we present the proof of the Power Rule, we introduce the Binomial Theorem.

Theorem. Binomial Theorem

Let a and b be real numbers and let n be a positive integer. Then

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \ldots + nab^{n-1} + b^n \\ &= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} a^{n-i}b^i\end{aligned}$$

where $\begin{pmatrix} n \\ i \end{pmatrix} = \dfrac{n!}{(n-i)!i!}$ and $i! = (i)(i-1)(i-2)\cdots(3)(2)(1)$.

Note. We can prove the Binomial Theorem using *Mathematical Induction.*

Proof of the Power Rule. By definition,

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{\Sigma_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^i - x^n}{h} \\ &= \lim_{h\to 0} \frac{x^n + \Sigma_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^i - x^n}{h}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\Sigma_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^{i}}{h} \\
&= \lim_{h\to 0} \frac{h\,\Sigma_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^{i-1}}{h} \\
&= \lim_{h\to 0} \sum_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^{i-1} \\
&= \lim_{h\to 0} nx^{n-1} + \sum_{i=2}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^{i-1} \\
&= nx^{n-1}.
\end{aligned}
$$

QED

Note. See page 136 for a proof of the Power Rule that doesn't (explicitly) use the Binomial Theorem.

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}[x^n] = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

Note. The state of the General Version of the Power Rule is a bit premature. In fact, even defining what it means to have an irrational exponent requires the use of exponential functions. None-the-less, the text states it now!

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}[cu] = c\frac{du}{dx}.$$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u+v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}[u+v] = \frac{du}{dx} + \frac{dv}{dx}.$$

Note. The proofs of the Derivative Constant Multiple Rule and the Derivative Sum Rule follow from the corresponding rules for limits (namely, the Constant Multiple Rule and the Sum Rule, respectively).

Corollary. (Page 144 number 71) If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, then $P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-1} + \cdots + 2a_2x + a_1$.

Example. Page 143 numbers 4, 12, and 34.

Note. We now differentiate an exponential function $f(x) = a^x$ where $a > 0$. By definition,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h} \\ &= \lim_{h\to 0} a^x \frac{a^h - 1}{h} \\ &= a^x \lim_{h\to 0} \frac{a^h - 1}{h}. \end{aligned}$$

We **claim without justification** that $\lim_{h\to 0} \frac{a^h - 1}{h}$ exists and is some number L_a dependent on a. (For a clean discussion of this result, see sections 7.2 and 7.3 of *Thomas Calculus*, Standard 11th Edition—notes are available online at `http://faculty.etsu.edu/gardnerr/1920/12/notes12.htm`. There is a version in this text in section 7.1.) With $x = 0$, we have $f'(0) = a^0 \lim_{h\to 0} \frac{a^h - 1}{h} = \lim_{h\to 0} \frac{a^h - 1}{h} = L_a$. We will see the precise value of L_a in section 3.7. Now $f'(0)$ is the slope of the graph of $y = a^x$ at $x = 0$. Motivated by Figure 3.11, we see that there is a value of a somewhere between 2 and 3 such that this slope is 0.

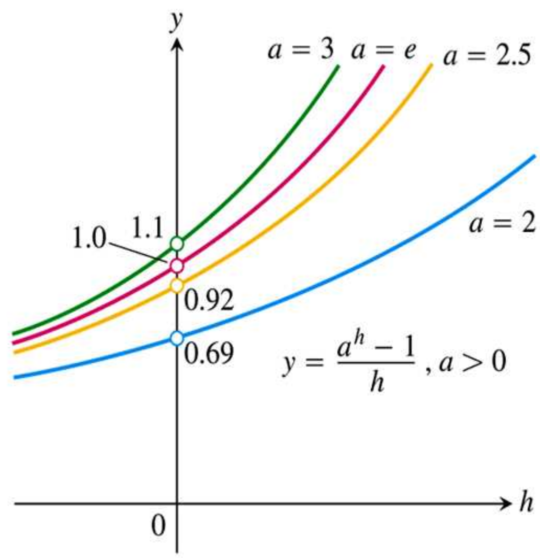

Figure 3.12, page 139

We define e to be the number for which the slope of the line tangent to $y = e^x$ is $m = 1$ at $x = 0$. That is, we define e such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. One can determine numerically (for a technique, see pages 183 and 184) that $e \approx 2.7182818284590459$. What is *natural* about the natural exponential function e^x is a *calculus* property—a *differentiation* property.

Theorem. Derivative of the Natural Exponential Function.

$$\frac{d}{dx}[e^x] = e^x.$$

Example. Differentiate $f(x) = x + 5e^x$.

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx} = [u']v + u[v'].$$

Proof. By definition we have:

$$\begin{aligned}
\frac{d}{dx}[uv] &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \left(u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h} \right) \\
&= \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\
&= u(x)[v'(x)] + [u'(x)]v(x).
\end{aligned}$$

where $\lim_{h\to 0} u(x+h) = u(x)$ since u is continuous at x by Theorem 1 of section 2.1. *QED*

Example. Differentiate $f(x) = (4x^3 - 5x^2 + 4)(7x^2 - x)$.

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} = \frac{[u']v - u[v']}{v^2}.$$

Proof. By definition we have:

$$\begin{aligned}
\frac{d}{dx}\left[\frac{u}{v}\right] &= \lim_{h\to 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \\
&= \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)} \\
&= \lim_{h\to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\
&= \lim_{x\to 0} \frac{v(x)\frac{u(x+h)-u(x)}{h} - u(x)\frac{v(x+h)-v(x)}{h}}{v(x+h)v(x)} \\
&= \frac{\lim_{h\to 0} v(x)\frac{u(x+h)-u(x)}{h} - \lim_{h\to 0} u(x)\frac{v(x+h)-v(x)}{h}}{\lim_{h\to 0} v(x+h)v(x)} \\
&= \frac{v(x)\lim_{h\to 0} \frac{u(x+h)-u(x)}{h} - u(x)\lim_{h\to 0} \frac{v(x+h)-v(x)}{h}}{v(x)\lim_{h\to 0} v(x+h)} \\
&= \frac{v(x)u'(x) - u(x)v'(x)}{v^2(x)}.
\end{aligned}$$

QED

Example. Page 143 number 20.

Example. Page 145 number 78, page 143 number 48.

Note. We will follow my "square brackets" notation as described in the handout.

Note. We can also calculate higher *order* derivatives:

$$y'' = \frac{d}{dx}[y'], y''' = \frac{d}{dx}[y''], y^{(4)} = \frac{d}{dx}[y'''], \ldots, y^{(n)} = \frac{d}{dx}[y^{(n-1)}].$$

Example. Page 143 number 42.