

Chapter 3. Differentiation

3.5 Derivatives of Trigonometric Functions

Recall. For all real numbers a and b ,

$$\sin(a + b) = \sin a \cos b + \cos a \sin b.$$

Theorem. Derivative of the Sine Function

$$\frac{d}{dx}[\sin x] = \cos x$$

Proof. Let $y = \sin x$. By definition we have

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \left(\cos x \cdot \frac{\sin h}{h} \right) \\ &= \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= \sin x \cdot 0 + \cos x \cdot 1 \\ &= \cos x. \end{aligned}$$

We have $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$ by the results in Section 2.4. *QED*

Example. Page 159 number 2.

Recall. For all real numbers a and b we have

$$\cos(a + b) = \cos a \cos b - \sin a \sin b.$$

Theorem. Derivative of the Cosine Function

$$\frac{d}{dx}[\cos x] = -\sin x$$

Proof. By definition we have

$$\begin{aligned} \frac{d}{dx}[\cos x] &= \lim_{h \rightarrow 0} \frac{\cos(x + h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \rightarrow 0} \sin x \cdot \frac{\sin h}{h} \\ &= \cos x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \end{aligned}$$

$$\begin{aligned} &= \cos x \cdot 0 - \sin x \cdot 1 \\ &= -\sin x. \end{aligned}$$

QED

Examples. Page 160 number 28, page 159 Example 5, and page 161 number 60.

Note. In summary, we have the following derivatives of the six trigonometric functions:

f	f'
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

Example. Page 160 number 40.