Chapter 3. Differentiation

3.6 The Chain Rule

Note. The **C**hain Rule allows us to differentiate **C**ompositions of functions.

Theorem 2. The Chain Rule.

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x))[g'(x)].$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Note. Every time we use the Chain Rule, we will insert a little arrow indicating that the Chain Rule "spits out" the derivative of the inner function in the composition:

$$(f \circ g)'(x) = f'(g(x))[g'(x)] \text{ and } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Note. The proof of the Chain Rule is rather complicated — see Section 3.11.

Note. If $f(u) = u^n$ where n is an integer, then

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[(g(x))^n] = ng(x)^{n-1}[g'(x)].$$

The text calls this the *Power Chain Rule*.

Examples. Page 167 number 8, Page 168 numbers 48 and 64.

Note. If $f(u) = e^u$ where $u = g(x)$ is a function of x, then

$$\frac{d}{dx}[e^u] = \frac{d}{dx}[e^{g(x)}] = e^{g(x)}[g'(x)].$$

Examples. Page 168 number 58, page 169 number 102