Chapter 3. Differentiation

3.7 Implicit Differentiation

Definition. The function $f(x)$ is *implicit* to the equation $F(x, y) = 0$ if the substitution $y = f(x)$ into the equation yields an identity.

Example. The functions $f(x) = \sqrt{1 - x^2}$ and $g(x) = -\sqrt{1 - x^2}$ are implicit to the equation $x^2 + y^2 = 1$. Can you find other functions implicit to this equation?

Note. If y is a function implicit to $F(x, y) = 0$, then we can generate an equation containing dy/dx by differentiating "implicitly." This follows by applying the Chain Rule.

Example. Suppose $y = f(x)$ is implicit to $x^2 + y^2 = 1$. Then differentiating implicitly:

$$\begin{aligned} \frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx}[1] \\ 2x + 2y\left[\frac{dy}{dx}\right] &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

Notice that dy/dx involves both x and y. This is because we cannot find the slope of a line tangent to the graph of $F(x, y) = 0$ without knowing the x and y coordinates of the point of tangency.

Example. Find the slope of the line tangent to $x^2 + y^2 = 1$ at $(x, y) = (\sqrt{2}/2, \sqrt{2}/2)$. Do the same for the point $(x, y) = (\sqrt{2}/2, -\sqrt{2}/2)$.

Definition. A line is *normal* to a curve at a point if it is perpendicular to the curve's tangent line. The line is called the *normal* to the curve at the point.

Examples. Page 174 numbers 16, 22, 38, and 42, Page 175 number 48.