Chapter 3. Differentiation

3.8. Derivatives of Inverse Functions and Logarithms

Note. Recall that the graph of a one-to-one function f and its inverse f^{-1} are mirror images of each other about the line $y = x$.

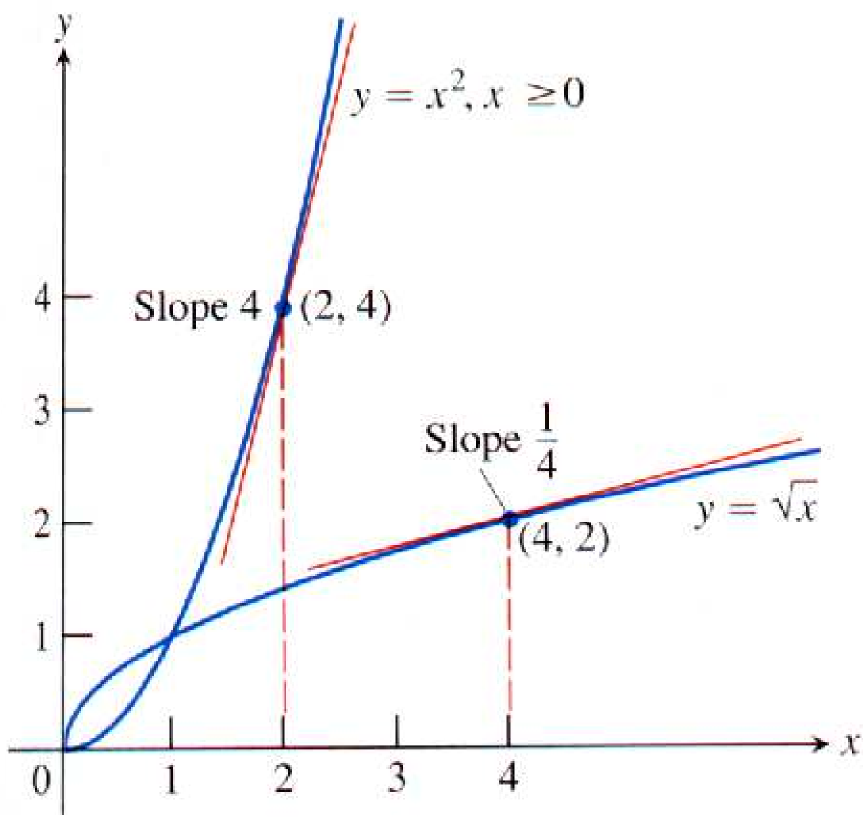

Figure 3.36 page 177

Theorem 3. The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

Proof. By definition of inverse function, $f^{-1}(f(x)) = x$ for all $x \in I$. Differentiating this equation, we have by the Chain Rule:

$$\frac{d}{dx}\left[f^{-1}(f(x))\right] = \frac{d}{dx}[x]$$

$$f^{-1\prime}(f(x))[f'(x)] = 1$$

$$f^{-1\prime}(f(x)) = \frac{1}{f'(x)}.$$

Plugging in $x = f^{-1}(b)$, we get the theorem. *Q.E.D.*

Example. Page 184 number 8.

Theorem. For $x > 0$ we have

$$\frac{d}{dx}\left[\ln x\right] = \frac{1}{x}.$$

If $u = u(x)$ is a differentiable function of x, then for all x such that $u(x) > 0$ we have

$$\frac{d}{dx}\left[\ln u\right] = \frac{d}{dx}\left[\ln u(x)\right] = \frac{1}{u}\left[\frac{du}{dx}\right] = \frac{1}{u(x)}\left[u'(x)\right].$$

Proof. We know that $f(x) = e^x$ is differentiable for all x, so we can apply Theorem 3 to find the derivative of $f^{-1}(x) = \ln x$:

$$\begin{aligned} \frac{d}{dx}[\ln x] = (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{e^{f^{-1}(x)}} \\ &= \frac{1}{e^{\ln x}} \\ &= \frac{1}{x}. \end{aligned}$$

By the Chain Rule

$$\frac{d}{dx}\left[\ln u(x)\right] = \frac{d}{du}\left[\ln u\right]\left[\frac{du}{dx}\right] = \frac{1}{u}\left[\frac{du}{dx}\right].$$

Q.E.D.

Note. We can apply the previous theorem to show that $\dfrac{d}{dx}[\ln |x|] = \dfrac{1}{x}$.

Recall. For any numbers $a > 0$ and for any real x, $a^x = e^{x \ln a}$.

Theorem. If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}[a^u] = a^u \ln a \left[\frac{du}{dx}\right].$$

Proof. First

$$\begin{aligned} \frac{d}{dx}[a^x] &= \frac{d}{dx}\left[e^{x \ln a}\right] \\ &= e^{x \ln a}\left[\frac{d}{dx}[x \ln a]\right] \\ &= a^x \ln a. \end{aligned}$$

Combining this result with the Chain Rule yields the theorem. *Q.E.D.*

Note. Notice that the previous theorem implies that $\frac{d}{dx}[a^x] = a^x \ln a$. With $a = e$, we have the special case $\frac{d}{dx}[e^x] = e^x(1) = e^x$. This is what is *natural* about e When you first meet the natural exponential and logarithmic functions in algebra, it is hard to understand what is NATURAL about them. That is because the "natural-ness" is a calculus property (namely this differentiation property).

Note. We saw in section 3.3 that $\dfrac{d}{dx}[a^x] = a^x \left(\lim\limits_{h \to 0} \dfrac{a^h - 1}{h} \right)$. We said then that the limit exists. We now see that the limit is $\lim\limits_{h \to 0} \dfrac{a^h - 1}{h} = \ln a$. In particular, for $a = e$, $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h} = \ln e = 1$.

Example. Page 185 number 70.

Definition. For any $a > 0$, $a \neq 1$, define $\log_a x = \dfrac{\ln x}{\ln a}$. (This is called the *change of base* formula. See page 45.)

Theorem. Differentiating a logarithm base a gives:

$$\frac{d}{dx}\left[\log_a u\right] = \frac{1}{\ln a}\frac{1}{u}\left[\frac{du}{dx}\right].$$

Proof. This follows easily:

$$\frac{d}{dx}\left[\log_a x\right] = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{\ln a}\frac{d}{dx}\left[\ln x\right] = \frac{1}{\ln a}\frac{1}{x}.$$

Combining this result with the Chain Rule gives the theorem. *Q.E.D.*

Examples. Page 185 numbers 74, and 80.

Note. We can, in fact, take the logarithm of a complicated function before differentiating it and then implicitly differentiate the result. This process is called *logarithmic differentiation*. It allows us to use the laws of logarithms instead of some of the complicated rules of differentiation.

Example. Page 185 number 90.

Definition. For any $x > 0$ and for any real number n, define $x^n = e^{n \ln x}$. (Here, we have finally formally defined what it means to exponentiate with irrational exponents.)

Theorem. General Power Rule for Derivatives.

For $x > 0$ and any real number n,

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

If $x < 0$, then the formula holds whenever the derivative x^n, and x^{n-1} all exist.

Proof. We have

$$\begin{aligned}\frac{d}{dx}[x^n] &= \frac{d}{dx}\left[e^{n\ln x}\right] \\ &= e^{n\ln x}\frac{d}{dx}[n\ln x] \text{ by the Chain Rule} \\ &= x^n\frac{n}{x} \\ &= nx^{n-1}.\end{aligned}$$

Q.E.D.

Example. Page 185 Example 72.

Theorem 4. The Number e as a Limit

We can find e as a limit:

$$e = \lim_{x\to 0}(1+x)^{1/x}.$$

Proof. Let $f(x) = \ln x$. Then $f'(x) = 1/x$ and $f'(1) = 1$. Now by the definition of derivative:

$$\begin{aligned}f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\frac{(1+x)-f(1)}{x} \\ &= \lim_{x\to 0}\frac{\ln(1+x)-\ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x) \\ &= \lim_{x\to 0}\ln(1+x)^{1/x} \\ &= \ln\left(\lim_{x\to 0}(1+x)^{1/x}\right) \text{ since } \ln x \text{ is continuous.}\end{aligned}$$

Therefore since $f'(1) = 1$ we have

$$\ln\left(\lim_{x\to 0}(1+x)^{1/x}\right) = 1.$$

Since $\ln e = 1$ and $\ln x$ is one-to-one,

$$\lim_{x\to 0}(1+x)^{1/x} = e.$$

Q.E.D.

Note. We can use the previous theorem to find that

$$e \approx 2.7\ 1828\ 1828\ 45\ 90\ 45\ 9.$$