Chapter 4. Applications of Derivatives

4.3 Monotonic Functions and The First Derivative Test

Definition. Let f be a function defined on an interval I. Then

1. f *increases* on I if for all points x_1 and x_2 in I,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$

2. f *decreases* on I if for all points x_1 and x_2 in I,

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

A function that is increasing or decreasing on I is called *monotonic* on I.

Corollary 3. The First Derivative Test for Increasing and Decreasing.

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b)

If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.

If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

Proof. Suppose $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ implies that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some c between x_1 and x_2. Since $x_2 - x_1 > 0$, then $f(x_2) - f(x_1)$ and $f'(c)$ are of the same sign. Therefore $f(x_2) > f(x_1)$ if f' is positive on (a, b), and $f(x_2) < f(x_1)$ if f' is negative on (a, b). *QED*

Example. Page 242 number 28a.

Note. First Derivative Test for Local Extrema.

At a critical point $x = c$,

1. f has a *local minimum* if f' changes from negative to positive at c

2. f has a *local maximum* if f' changes from positive to negative at c

3. f has *no local extreme* if f' has the sign on both sides of c.

Example. Page 242 number 28b.

Example. Page 242 number 68.