Chapter 5. Integration

5.3 The Definite Integral

Definition. Let f be a function defined on a closed interval $[a, b]$. We say that a number I is the *definite integral of* f *over* $[a, b]$ and that J is the limit of the Riemann sums if the following condition is satisfied: Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of $c_k \in [x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\, \Delta x_k - J \right| < \epsilon.$$

We denote $J = \int_a^b f(x)\, dx$ and say that f is *integrable* on $[a, b]$.

Note. We can also say:

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\, \Delta x_k = \int_a^b f(x)\, dx.$$

Example. Page 322 number 6.

Note. When we deal with applications of integration, we will often think of definite integrals as sums. Notice, however, that strictly speaking they are not sums, but they are *limits* of sums.

Note. We have now introduced three ideas, each different from the other, but each related to the other (as we will see when we state the Fundamental Theorem of Calculus). We have:

Name of Object	**Type of Object**
ANTIDERIVATIVE	*FUNCTION*
INDEFINITE INTEGRAL	*COLLECTION* or *SET*
DEFINITE INTEGRAL	*NUMBER*

Antiderivatives and indefinite integrals are related by the fact that the indefinite integral of a function f is the set of all antiderivatives of f. The Fundamental Theorem of Calculus, to be seen in the next section, will relate antiderivatives and definite integrals (and therefore will relate definite and indefinite integrals).

Theorem 1. Integrability of Continuous Functions.

If a function f is continuous on an interval $[a,b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a,b]$.

Theorem 2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a,b]$. Then:

1. *Order of Integration*: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (this in fact is a definition)

2. *Zero Width Interval*: $\int_a^a f(x)\,dx = 0$ (this too is a definition)

3. *Constant Multiple*: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

4. *Sum and Difference*: $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity*: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality*: If $\max f$ and $\min f$ are the maximum and minimum values of f on $[a,b]$, then

$$\min f \cdot (b-a) \leq \int_a^b f(x)\,dx \leq \max f \cdot (b-a).$$

7. *Domination*: $f(x) \geq g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Example. Page 322 number 10.

Note. If we partition $[a, b]$ into n pieces of equal length $(b-a)/n$, then the partition if *regular*. We can then evaluate the limit $\|P\| \to 0$ by letting $n \to \infty$. This can be used to evaluate definite integrals. If we do so, then these formulas are useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Example. Use a regular partition of $[0, 1]$ with $c_k = x_k$ to evaluate $\int_0^1 x^2\,dx$. Notice that in page 320 Example 4b it is shown that $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$, and in page 323 number 65 it is shown that $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$.

Example. Page 322 number 36.

Definition. If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the *area under the curve $y = f(x)$ from a to b* is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

Example. Page 322 number 18.

Definition. If f is integrable on $[a, b]$, then its *average (mean) value* of $[a, b]$ is

$$\operatorname{av}(f) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Example. Page 323 number 62.

Example. Page 323 number 71.