Chapter 2. Limits and Continuity

2.2. Calculating Limits Using the Limit Laws

Theorem 1. Limit Rules.

If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x\to c}(f(x) + g(x)) = L + M$.

2. *Difference Rule*: $\lim_{x\to c}(f(x) - g(x)) = L - M$.

3. *Product Rule*: $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$.

4. *Constant Multiple Rule*: $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$.

5. *Quotient Rule*: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number **AND** $L > 0$ when s is even.

Example. Page 84 number 38.

Theorem 2. Limits of Polynomials Can Be Found by Substitution.

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

Theorem 3. Limits of Rational Functions Can Be Found by Substituting IF the Limit of the Denominator Is Not Zero.

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example. Page 83 numbers 4 and 10.

Theorem. Dr. Bob's Theorem. (NOT IN 11TH EDITION!)

If $f(x) = g(x)$ for all x in an open interval containing c, except possibly c itself, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

provided these limits exist.

Note. We have to be careful in our dealings with functions! Notice that $f(x) = \dfrac{x(x-1)}{x-1}$ and $g(x) = x$ are **NOT** the same functions! They do not even have the same domains. Therefore we cannot in general say $\dfrac{x(x-1)}{x-1} = x$. However, this equality holds if x lies in the domains of the functions. We *can* say:

$$\frac{x(x-1)}{x-1} = x \textbf{ IF } x \neq 1.$$

We can also say $f(x) = g(x)$ **IF** $x \neq 1$. If we are concerned with limits as x approaches 1, then from the definition, x **IS NOT EQUAL TO 1** (but near 1). Therefore we can say $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x)$. We have not said that the functions are equal, but that their limits are.

Example. Page 83 numbers 28 and 32.

Theorem 4. Sandwich Theorem.

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

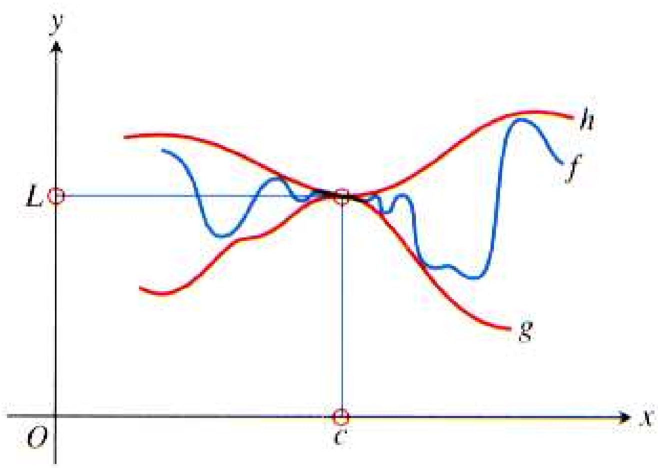

Figure 2.9, page 82

Example. Page 84 number 52.

Example. Page 85 number 55.