Chapter 2. Limits and Continuity

2.3 The Precise Definition of a Limit

Definition. Formal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the *limit* L *as* x *approaches* x_0 and write $\lim_{x \to x_0} f(x) = L$, if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

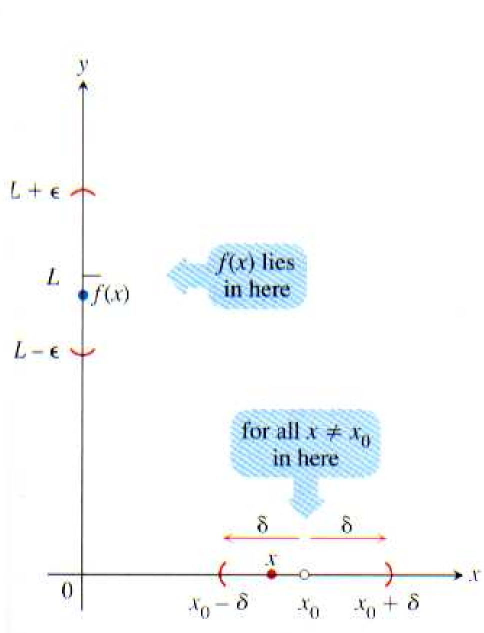

Figure 1.11 from Edition 10, page 93

Example. Prove for $f(x) = mx + b$, $m \neq 0$, that $\lim_{x \to a} f(x) = f(a)$.

Examples. Page 92 number 12, page 93 number 20.

Theorem 1. Sum Rule. If

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

then $\lim_{x \to c}(f(x) + g(x)) = L + M$.

Proof. We wish to prove $\lim_{x \to c}(f(x) + g(x)) = L + M$ under the assumptions $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Let $\epsilon > 0$ be given. Then $\epsilon/2 > 0$ and there exists $\delta_1 > 0$ such that for all x with $0 < |x - c| < \delta_1$ we have $|f(x) - L| < \epsilon/2$. Similarly, there exists $\delta_2 > 0$ such that for all x with $0 < |x - c| < \delta_2$ we have $|g(x) - M| < \epsilon/2$. Therefore we choose $\delta = \min\{\delta_1, \delta_2\}$. Then for $0 < |x - c| < \delta$ we have

$$\begin{aligned}
|(f(x) + g(x)) - (L + M)| &\leq |(f(x) - L) + (g(x) - M)| \\
&\leq |f(x) - L| + |g(x) - M| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This proves the result. *Q.E.D.*

Example. Page 94 number 58.