Chapter 2. Limits and Continuity

2.4 One-Sided Limits and Limits at Infinity

Definition. Informal Definition of Right-Hand and Left-Hand Limits.

Let $f(x)$ be defined on an interval (a, b), where $a < b$. If $f(x)$ approaches arbitrarily close to L as x approaches a from within that interval, then we say that f has *right-hand limit* L at a, and write

$$\lim_{x \to a^+} f(x) = L.$$

Let $f(x)$ be defined on an interval (c, a), where $c < a$. If $f(x)$ approaches arbitrarily close to M as x approaches a from within the interval (c, a), then we say that f has *left-hand limit* M at a, and we write

$$\lim_{x \to a^-} f(x) = M.$$

Definition. Formal Definitions of One Sided Limits.

We say that $f(x)$ has *right-hand limit L at x_0*, and write

$$\lim_{x\to x_0^+} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that $f(x)$ has *left-hand limit L at x_0*, and write

$$\lim_{x\to x_0^-} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Example. Consider limits as x approaches -1 and $+1$ for $f(x) = \sqrt{1-x^2}$.

Example. (Example 3 page 98) Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Proof. Let $\epsilon > 0$ be given. We have $x_0 = 0$ and $L = 0$. Choose $\delta = \epsilon^2$. Then if $0 < x - x_0 = x - 0 = x < \delta = \epsilon^2$, we have $\sqrt{x} < \epsilon$, or $|\sqrt{x} - 0| < \epsilon$, or $|f(x) - L| < \epsilon$. Therefore, the result holds. *QED*

Theorem 6. Relation Between One-Sided and Two-Sided Limits

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Example. Page 107 number 10.

Example. Page 109 number 70.

Theorem 7.

For θ in radians,

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

Proof. Suppose first that θ is positive and less than $\pi/2$. Consider the picture:

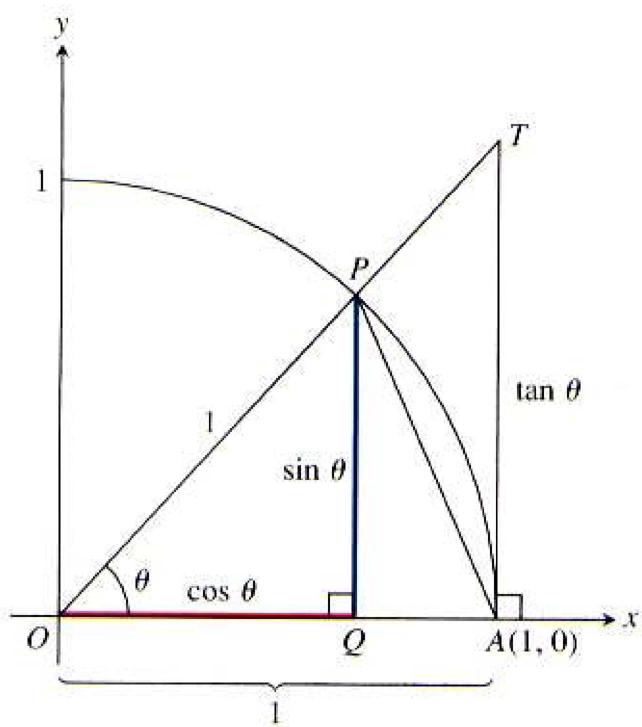

Figure 2.30, page 99

Notice that

$$\text{Area } \Delta OAP < \text{area sector } OAP < \text{area } \Delta OAT.$$

We can express these areas in terms of θ as follows:

$$\text{Area } \Delta OAP = \tfrac{1}{2} \text{ base} \times \text{height} = \tfrac{1}{2}(1)(\sin\theta) = \tfrac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(1)^2\theta = \tfrac{\theta}{2}$$

$$\text{Area } \Delta OAT = \tfrac{1}{2} \text{ base} \times \text{height} = \tfrac{1}{2}(1)(\tan\theta) = \tfrac{1}{2}\tan\theta.$$

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the positive number $(1/2)\sin\theta$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Since $\lim_{\theta\to 0^+} \cos\theta = 1$, the Sandwich Theorem gives

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

Since $\sin\theta$ and θ are both odd functions, $f(\theta) = \dfrac{\sin\theta}{\theta}$ is an even function and hence $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin\theta}{\theta}$. Therefore

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta},$$

so $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ by Theorem 4. *QED*

Example. Page 100 example 5a: Show that $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$.

Solution. We use the trig identity $\cos h = 1 - 2\sin^2(h/2)$ (a half-angle identity). First, we have

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} -\frac{2\sin^2(h/2)}{h}.$$

Now, replacing $h/2$ with θ we get

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} -\frac{2\sin^2(h/2)}{h} = -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \sin\theta = -(1)(0) = 0.$$

QED

Definition. Formal Definition of Limits at Infinity.

1. We say that $f(x)$ has the *limit L as x approaches infinity* and we write

$$\lim_{x \to +\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the *limit L as x approaches negative infinity* and we write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Example. Page 108 number 34.

Definition. Informal Definition of Limits Involving Infinity.

1. We say that $f(x)$ has the *limit L as x approaches infinity* and write

$$\lim_{x \to +\infty} f(x) = L$$

if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L.

2. We say that $f(x)$ has the *limit L as x approaches negative infinity* and write

$$\lim_{x \to -\infty} f(x) = L$$

if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L.

Example. Example 6 page 102. Show that $\lim_{x\to\infty} \frac{1}{x} = 0$.

Solution. Let $\epsilon > 0$ be given. We must find a number M such that for all

$$x > M \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number (see the figure below). This proves $\lim_{x\to\infty} \frac{1}{x} = 0$. We can similarly prove that $\lim_{x\to-\infty} \frac{1}{x} = 0$. *QED*

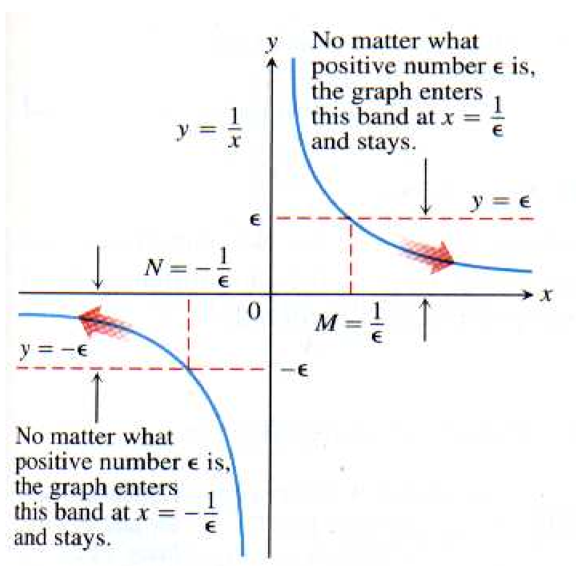

Figure 2.32, page 102

Theorem 8. Rules for Limits as $x \to \pm\infty$.

If L, M, and k are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\pm\infty} = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x\to\pm\infty}(f(x) + g(x)) = L + M$

2. *Difference Rule*: $\lim_{x\to\pm\infty}(f(x) - g(x)) = L - M$

3. *Product Rule*: $\lim_{x\to\pm\infty}(f(x) \cdot g(x)) = L \cdot M$

4. *Constant Multiple Rule*: $\lim_{x\to\pm\infty}(k \cdot f(x)) = k \cdot L$

5. *Quotient Rule*: $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, M \neq 0$

6. *Power Rule*: If r and s are integers with no common factors and $s \neq 0$, then

$$\lim_{x\to\pm\infty}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number **AND** $L > 0$ when s is even.

Example. Page 108 number 52 and number 48.

Definition. Horizontal Asymptote.

A line $y = b$ is a *horizontal asymptote* of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

Example. Page 118 number 32, find the horizontal asymptotes.

Definition. Oblique Asymptotes.

If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an *oblique asymptote* (or *slant asymptote*). The asymptote is found by dividing the denominator into the numerator to express the function as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

Example. Page 118 number 36, find the slant asymptotes.

Example. Page 109 number 74.