Chapter 2. Limits and Continuity

2.5 Infinite Limits and Vertical Asymptotes

Definition. Infinity, Negative Infinity as Limits

1. We say that $f(x)$ *approaches infinity as* x *approaches* x_0, and we write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

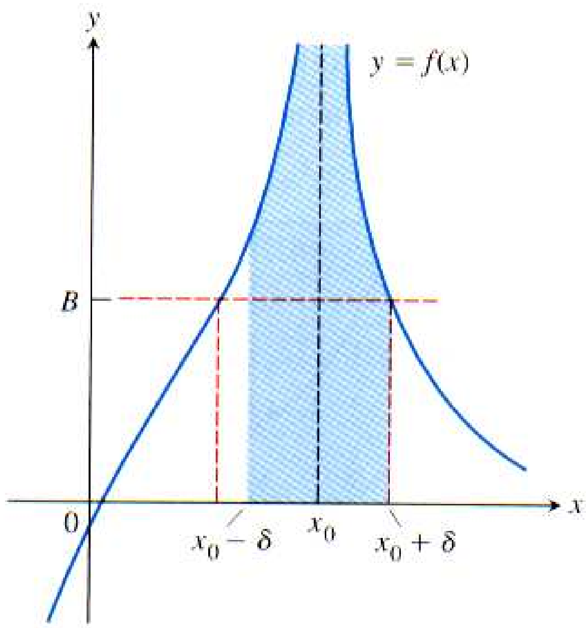
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B.$$

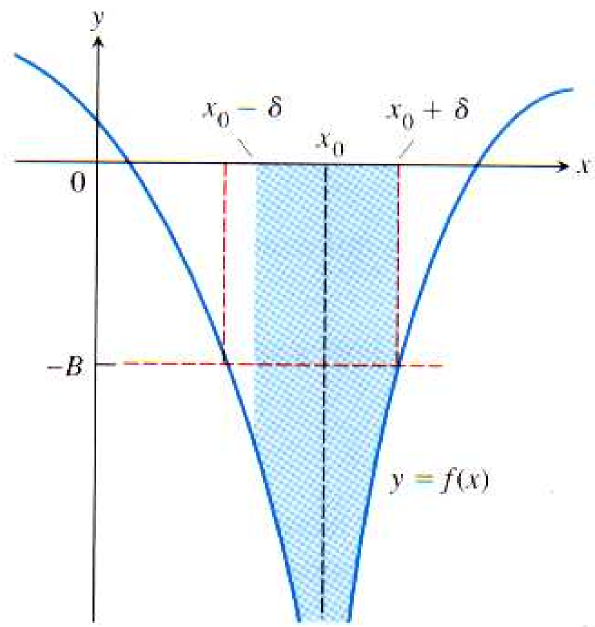
2. We say that $f(x)$ *approaches negative infinity as* x *approaches* x_0, and we write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B.$$





Figures 2.40 and 2.41, 10th Edition.

Note. Informally, $\lim_{x \to x_0} f(x) = \infty$ if $f(x)$ can be made arbitrarily large by making x sufficiently close to x_0 (and similarly for f approaching negative infinity). We can also define one-sided infinite limits in an analogous manner (see page 118 number 51).

Definition. Vertical Asymptotes.

A line $x = a$ is a *vertical asymptote* of the graph if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty.$$

Note. Recall that we look for the vertical asymptotes of a rational function where the denominator is zero (though just because the denominator has zero at a point, the function does not *necessarily* have a vertical asymptote at that point). We make things more precise in the following result:

Dr. Bob's Second Theorem. Let $f(x) = \dfrac{p(x)}{q(x)}$. Suppose $\lim_{x \to x_0} p(x) = L \neq 0$, $\lim_{x \to x_0} q(x) = 0$, and $q(x)$ is of the same sign in some open interval containing x_0. Then $\lim_{x \to x_0} f(x) = \pm\infty$. We can say something similar for one-sided limits.

Note. We can simplify Dr. Bob's Second Theorem by applying it to rational functions. It then becomes: "Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function. Suppose $\lim_{x \to x_0^+} p(x) = L \neq 0$ and $\lim_{x \to x_0^+} q(x) = 0$. Then $\lim_{x \to x_0^+} f(x) = \pm\infty$." We can say something similar for limits from the left and for two-sided limits.

Examples. Page 117 numbers 18. Page 118 numbers 36 (again), 44 and 54.