# **Chapter 2.** Limits and Continuity

## **2.6** Continuity

**Definition. Continuity at a Point.**

**Interior Point**: A function $y = f(x)$ is *continuous at an interior point* $c$ of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

**Endpoint**: A function $y = f(x)$ is *continuous at a left endpoint* $a$ or is *continuous at a right endpoint* $b$ of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

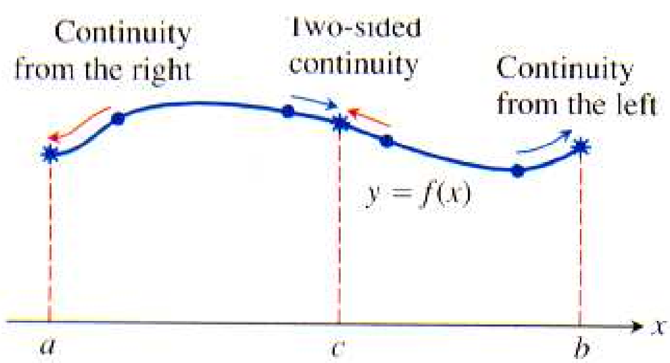


Figure 2.55, page 120.

**Note.** If a function is continuous at all interior points of its domain and the domain is an interval, then the function can be "drawn without picking up your pencil."

**Example.** Page 129 number 4.

**Continuity Test.**

A function $f(x)$ is continuous at an interior point of the domain of $f$, $x = c$, if and only if it meets the following three conditions:

**1.** $f(c)$ exists,

**2.** $\lim_{x \to c} f(x)$ exists, and

**3.** $\lim_{x \to c} f(x) = f(c)$.

**Note.** Polynomials, rational functions, and the six trigonometric functions are continuous at every point of their domains.

**Example.** Consider the piecewise defined function

$$f(x) = \begin{cases} x & \text{if } x \in (-\infty, 0) \\ 0 & \text{if } x = 0 \\ x^2 & \text{if } x \in (0, \infty) \end{cases}.$$

Is $f$ continuous at $x = 0$?

**Definition.** A function $f$ has a *removable discontinuity* at $x = a$ if $f(a)$ can be redefined in such a way that $f$ is continuous at $a$. $f$ has a *jump discontinuity* at $x = a$ if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist (as finite numbers) and are different. The book also gives examples of an *infinite discontinuity* and an *oscillating discontinuity* (see page 122).

**Example.** Discuss the discontinuities of $f(x) = \dfrac{|x|}{x}$ and $g(x) = \text{int } x$.

**Theorem 9. Properties of Continuous Functions**

If the functions $f$ and $g$ are continuous at $x = c$, then the following combinations are continuous at $x = c$.

**1.** *Sums*: $f + g$

**2.** *Differences*: $f - g$

**3.** *Products*: $f \cdot g$

**4.** *Constant Multiples*: $k \cdot f$, for any number $k$

**5.** *Quotients*: $f/g$, provided $g(c) \neq 0$.

**Theorem 10. Composite of Continuous Functions**

If $f$ is continuous at $c$ and $g$ is continuous at $f(c)$, then the composite $g \circ f$ is continuous at $c$.

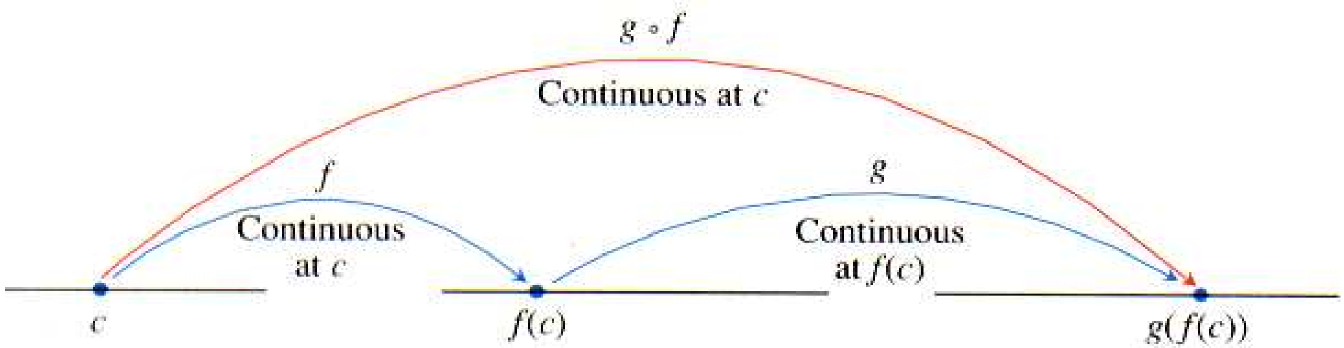


Figure 2.61, page 124.

**Example.** Page 129 number 30.

**Note.** If a function has a removable discontinuity at a point, then we can redefine the function at that point in such a way as to create a new function which *is* continuous at that point. This new function is called a *continuous extension* of the original function.

**Example.** Page 130 number 36.

**Theorem 11. The Intermediate Value Theorem for Continuous Functions**

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if $y_0$ is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $c$ in $[a, b]$.

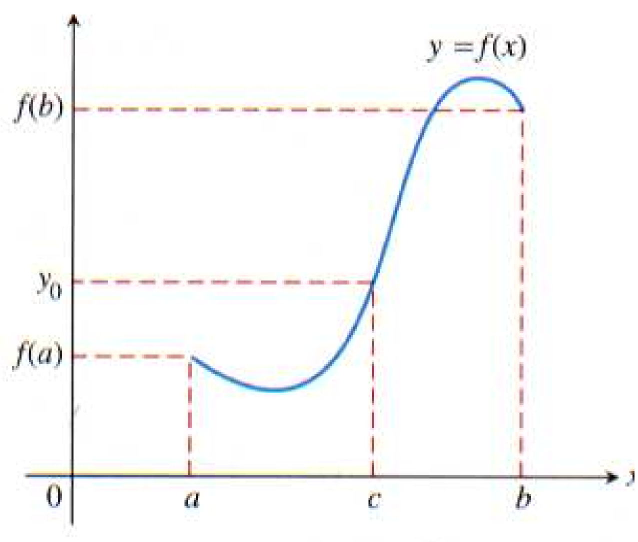


Figure from page 127.

**Examples.** Page 130 number 49a and number 58.