Chapter 3. Differentiation

3.1 The Derivative as a Function

Definition. Derivative Function.

The *derivative* of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Note. Motivated by Section 2.7, we see that $f'(x)$ is the slope of the line tangent to $y = f(x)$ as a function of x.

Note. There are a number of ways to denote the derivative of $y = f(x)$:

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}[f].$$

Examples. Example 2(a) page 146, page 153 numbers 16 and 30.

Note. We can also study "one-sided derviatives" at a point defined as follows:

$$\text{Right-hand derivative at } a: \quad \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$
$$\text{Left-hand derivative at } b: \quad \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$$

Example. Page 154 number 38.

Theorem 1. Differentiability Implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof. By definition, we need to show that $\lim_{x\to c} f(x) = f(c)$, or equivalently that $\lim_{h\to 0} f(c+h) = f(c)$. Then

$$\begin{aligned}
\lim_{h\to 0} f(c+h) &= \lim_{h\to 0}\left(f(c) + \frac{f(c+h)-f(c)}{h}\cdot h\right)\\
&= \lim_{h\to 0} f(c) + \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\cdot \lim_{h\to 0} h\\
&= f(c) + f'(c)\cdot 0\\
&= f(c).
\end{aligned}$$

Therefore f is continuous at $x = c$. *QED*

Example. Page 155 number 44.

Theorem 2. Intermediate Value Property of Derivatives

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

Example. Page 155 number 52.