Chapter 3. Differentiation

3.2 Differentiation Rules for Polynomials, Exponentials, Products and Quotients

Rule 1. Derivative of a Constant Function.

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}[c] = 0.$$

Proof. From the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to o} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

QED

Rule 2. Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Note. Before we present the proof of the Power Rule, we introduce the Binomial Theorem.

Theorem. Binomial Theorem

Let a and b be real numbers and let n be a positive integer. Then

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \ldots + nab^{n-1} + b^n \\ &= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} a^{n-i}b^i\end{aligned}$$

where $\begin{pmatrix} n \\ i \end{pmatrix} = \dfrac{n!}{(n-i)!i!}$ and $i! = (i)(i-1)(i-2)\cdots(3)(2)(1)$.

Note. We can prove the Binomial Theorem using *Mathematical Induction.*

Proof of the Power Rule. By definition,

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{\sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^i - x^n}{h} \\ &= \lim_{h\to 0} \frac{x^n + \sum_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^i - x^n}{h}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\Sigma_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^{i}}{h} \\
&= \lim_{h\to 0} \frac{h\,\Sigma_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^{i-1}}{h} \\
&= \lim_{h\to 0} \sum_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^{i-1} \\
&= \lim_{h\to 0} nx^{n-1} + \sum_{i=2}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^{i-1} \\
&= nx^{n-1}.
\end{aligned}
$$

QED

Note. See page 157 for a proof of the Power Rule that doesn't (explicitly) use the Binomial Theorem.

Rule 3. Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}[cu] = c\frac{du}{dx}.$$

Rule 4. Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}[u+v] = \frac{du}{dx} + \frac{dv}{dx}.$$

Note. The proofs of Rules 3 and 4 follow from the corresponding rules for limits (namely, the Constant Multiple Rule and the Sum Rule, respectively).

Corollary. If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, then $P'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-1} + \cdots + 2a_2 x + a_1$.

Note. We now differentiate an exponential function $f(x) = a^x$ where $a > 0$. By definition,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h} \end{aligned}$$

$$= \lim_{h \to 0} a^x \frac{a^h - 1}{h}$$
$$= a^x \lim_{h \to 0} \frac{a^h - 1}{h}.$$

We **claim without justification** that $\lim_{h \to 0} \frac{a^h - 1}{h}$ exists and is some number L_a dependent on a. (For a clean discussion of this result, see sections 7.2 and 7.3 of *Thomas Calculus*, Standard 11th Edition—notes are available online at `http://www.etsu.edu/math/gardner/1920/11/notes11.htm`. There is a version in this text in section 7.1.) With $x = 0$, we have $f'(0) = a^0 \lim_{h \to 0} \frac{a^h - 1}{h} = \lim_{h \to 0} \frac{a^h - 1}{h} = L_a$. We will see the precise value of L_a in section 3.7. Now $f'(0)$ is the slope of the graph of $y = a^x$ at $x = 0$. Motivated by Figure 3.11, we see that there is a value of a somewhere between 2 and 3 such that this slope is 0.

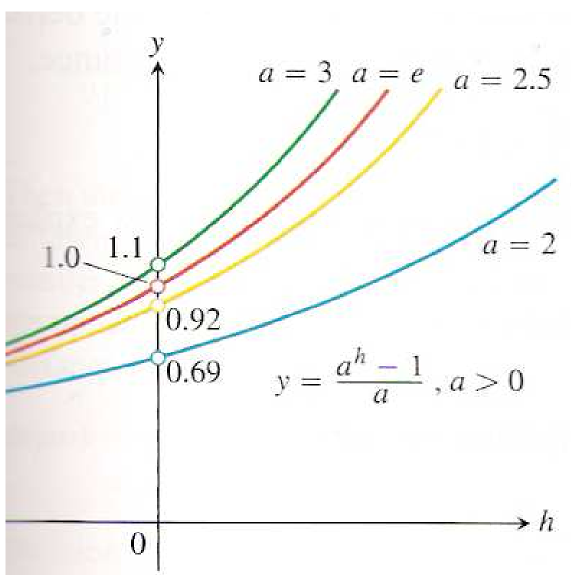

Figure 3.11, page 161

We define e to be the number for which the slope of the line tangent to $y = e^x$ is $m = 1$ at $x = 0$. That is, we define e such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. One can determine numerically (for a technique, see page 220) that $e \approx 2.718281828459045$9. What is *natural* about the natural exponential function e^x is a *calculus* property—a *differentiation* property.

Theorem. Derivative of the Natural Exponential Function.

$$\frac{d}{dx}[e^x] = e^x.$$

Example. Differentiate $f(x) = x + 5e^x$.

Note. We can also calculate higher *order* derivatives:

$$y'' = \frac{d}{dx}[y'], y''' = \frac{d}{dx}[y''], y^{(4)} = \frac{d}{dx}[y'''], \ldots, y^{(n)} = \frac{d}{dx}[y^{(n-1)}].$$

Example. Page 167 number 30.

Rule 5. Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx} = [u']v + u[v'].$$

Proof. By definition we have:

$$\begin{aligned}
\frac{d}{dx}[uv] &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \left(u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h} \right) \\
&= \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\
&= u(x)[v'(x)] + [u'(x)]v(x).
\end{aligned}$$

where $\lim_{h\to 0} u(x+h) = u(x)$ since u is continuous at x by Theorem 1 of section 2.1. *QED*

Example. Differentiate $f(x) = (4x^3 - 5x^2 + 4)(7x^2 - x)$.

Rule 6. Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} = \frac{[u']v - u[v']}{v^2}.$$

Proof. By definition we have:

$$\begin{aligned}
\frac{d}{dx}\left[\frac{u}{v}\right] &= \lim_{h\to 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \\
&= \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)} \\
&= \lim_{h\to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\
&= \lim_{x\to 0} \frac{v(x)\frac{u(x+h)-u(x)}{h} - u(x)\frac{v(x+h)-v(x)}{h}}{v(x+h)v(x)} \\
&= \frac{\lim_{h\to 0} v(x)\frac{u(x+h)-u(x)}{h} - \lim_{h\to 0} u(x)\frac{v(x+h)-v(x)}{h}}{\lim_{h\to 0} v(x+h)v(x)} \\
&= \frac{v(x)\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} - u(x)\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}}{v(x)\lim_{h\to 0} v(x+h)} \\
&= \frac{v(x)u'(x) - u(x)v'(x)}{v^2(x)}.
\end{aligned}$$

QED

Example. Page 167 number 20.

Rule 7. Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Proof. If n is a negative integer, then $-n$ is a positive integer and we can use the Power Rule for Nonnegative Integers to differentiate x^{-n}. So

$$\begin{aligned}\frac{d}{dx}[x^n] &= \frac{d}{dx}\left[\frac{1}{x^{-n}}\right] \\ &= \frac{\frac{d}{dx}[1](x^{-n}) - (1)\frac{d}{dx}[x^{-n}]}{(x^{-n})^2} \\ &= \frac{[0](x^{-n}) - (1)[(-n)x^{-n-1}]}{x^{-2n}} \\ &= nx^{n-1}.\end{aligned}$$

QED

Note. We have now established that $\frac{d}{dx}[x^n] = nx^{n-1}$ for all integers n. We will eventually see that this is the way x^n is differentiated for all real numbers n, but we have not even defined what it means to raise a real number to an *irrational* number! We will take care of this when we define the natural logarithm and exponential functions.

Example. Page 169 number 56, page 167 number 34.

Note. We will follow my "square brackets" notation as described in the handout.