Chapter 3. Differentiation

3.4 Derivatives of Trigonometric Functions

Recall. For all real numbers a and b,

$$\sin(a+b) = \sin a \cos b + \cos a \sin b.$$

Theorem. Derivative of the Sine Function

$$\frac{d}{dx}[\sin x] = \cos x$$

Proof. Let $y = \sin x$. By definition we have

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \left(\cos x \cdot \frac{\sin h}{h}\right) \\
&= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x.
\end{aligned}$$

We have $\lim_{h\to 0}\dfrac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0}\dfrac{\sin h}{h} = 1$ by the results in Section 2.4. *QED*

Example. Page 186 number 2.

Recall. For all real numbers a and b we have

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

Theorem. Derivative of the Cosine Function

$$\frac{d}{dx}[\cos x] = -\sin x$$

Proof. By definition we have

$$\begin{aligned}
\frac{d}{dx}[\cos x] &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{(\cos x\cos h - \sin x \sin h) - \cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0}\cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0}\sin x \cdot \frac{\sin h}{h} \\
&= \cos x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0}\frac{\sin h}{h}
\end{aligned}$$

$$= \cos x \cdot 0 - \sin x \cdot 1$$

$$= -\sin x.$$

QED

Examples. Page 186 number 22, page 185 Example 5, and page 187 number 50.

Note. In summary, we have the following derivatives of the six trigonometric functions:

f	f'
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

Example. Page 186 number 32.