Chapter 3. Differentiation

3.5 The Chain Rule and Parametric Equations

Theorem 3. The Chain Rule.

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x))[g'(x)].$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Note. The proof of the Chain Rule is rather complicated — see Section 3.8.

Note. If $f(u) = u^n$ where n is an integer, then

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[(g(x))^n] = ng(x)^{n-1}[g'(x)].$$

Examples. Page 199 numbers 8, 43, and 54, page 200 number 64.

Definition. A curve is given *parametrically* if its graph is determined by graphing $(x(t), y(t))$ for t (the *parameter*) ranging over some interval.

Example. $x(t) = \cos t$, $y(t) = \sin t$ for $t \in [0, 2\pi)$ parametrically determine the unit circle $x^2 + y^2 = 1$.

Note. Parametric Formula for dy/dx and d^2y/dx^2.
If $x = x(t)$ and $y = y(t)$ where $x(t)$ and $y(t)$ are differentiable with $x'(t) \neq 0$ then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \text{ and } \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

Examples. Pages 201 number 108, page 202 number 114.