Chapter 3. Differentiation

3.6 Implicit Differentiation

Definition. The function $f(x)$ is *implicit* to the equation $F(x, y) = 0$ if the substitution $y = f(x)$ into the equation yields an identity.

Example. The functions $f(x) = \sqrt{1 - x^2}$ and $g(x) = -\sqrt{1 - x^2}$ are implicit to the equation $x^2 + y^2 = 1$. Can you find other functions implicit to this equation?

Note. If y is a function implicit to $F(x, y) = 0$, then we can generate an equation containing dy/dx by differentiating "implicitly." This follows by applying the Chain Rule.

Example. Suppose $y = f(x)$ is implicit to $x^2 + y^2 = 1$. Then differentiating implicity:

$$\begin{aligned} \frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx}[1] \\ 2x + 2y\left[\frac{dy}{dx}\right] &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

Notice that dy/dx involves both x and y. This is because we cannot find the slope of a line tangent to the graph of $F(x, y) = 0$ without knowing the x and y coordinates of the point of tangency.

Example. Find the slope of the line tangent to $x^2 + y^2 = 1$ at $(x, y) = (\sqrt{2}/2, \sqrt{2}/2)$. Do the same for the point $(x, y) = (\sqrt{2}/2, -\sqrt{2}/2)$.

Definition. A line is *normal* to a curve at a point if it is perpendicular to the curve's tangent line. The line is called the *normal* to the curve at the point.

Examples. Page 209 number 32, page 210 number 54, and page 211 number 70.

Theorem 4. Power Rule for Rational Powers.

If n is a rational number, then x^n is differentiable at every interior point of the domain of x^{n-1}, and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Proof. Let p and q be integers with $q > 0$ and suppose that $y = \sqrt[q]{x^p} = x^{p/q}$. Then $y^q = x^p$. By the Power Rule for Integer Exponents,

$$\begin{aligned} \frac{d}{dx}\left[y^q\right] &= \frac{d}{dx}\left[x^p\right] \\ qy^{q-1}\left[\frac{dy}{dx}\right] &= px^{p-1}. \end{aligned}$$

If $y \neq 0$, then we can solve for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q}x^{(p/q)-1}.$$

QED