Chapter 4. Applications of Derivatives

4.2 The Mean Value Theorem

Theorem 3. Rolle's Theorem.

Suppose that $y = f(x)$ is continuous at every point of $[a, b]$ and differentiable at every point of (a, b). If $f(a) = f(b) = 0$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Proof. Since f is continuous by hypothesis, f assumes an absolute maximum and minimum for $x \in [a, b]$ by Theorem 1 (the Extreme Value Theorem). These extrema occur only

1. at interior points where f' is zero

2. at interior points where f' does not exist

3. at the endpoints of the function's domain, a and b.

Since we have hypothesized that f is differentiable on (a, b), then Option 2 is not possible.

In the event of Option 1, the point at which an extreme occurs, say c, must satisfy $f'(c) = 0$ by Theorem 2 of Section 3.1 (Local Extreme Values). Therefore the theorem holds.

In the event of Option 3, the maximum and minimum occur at the endpoints a and b (where f is 0) and so f must be a constant of 0 throughout the interval. Therefore $f'(x) = 0$ for all $x \in (a, b)$, by Rule 1 page 156, and the theorem holds. *QED*

Example. Page 284 number 54.

Theorem 4. The Mean Value Theorem

Suppose that $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval (a, b). Then there is at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

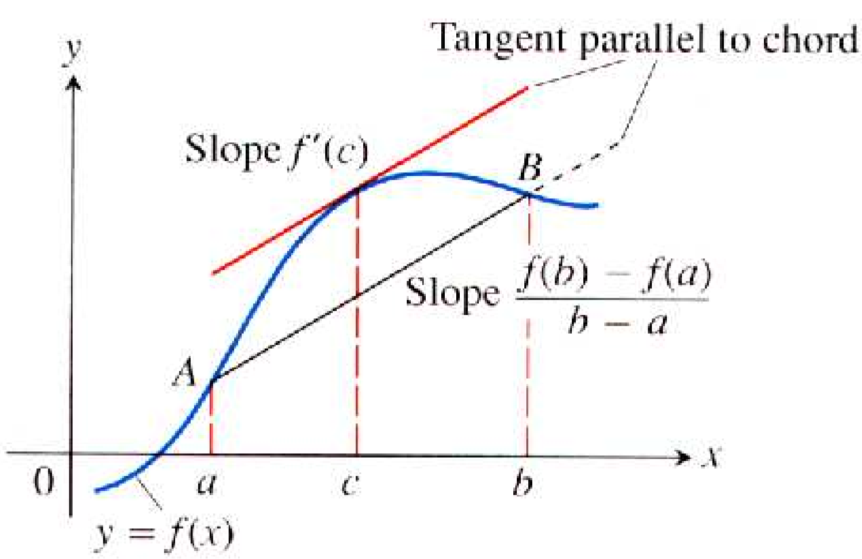

Figure 4.14, page 277

Examples. Page 282 number 2, page 284 numbers 46 and 59.

Corollary 1. Functions with Zero Derivatives Are Constant Functions.
If $f'(x) = 0$ at each point of an interval I, then $f(x) = k$ for all $x \in I$, where k is a constant.

Note. Corollary 1 is the *converse* of Rule 1 from page 149.

Corollary 2. Functions with the Same Derivative Differ by a Constant
If $f'(x) = g'(x)$ at each point of an interval (a, b), then there exists a constant k such that $f(x) = g(x) + k$ for all $x \in (a, b)$.

Proof. Consider the function $h(x) = f(x) - g(x)$. Under our hypothesis, $h(x)$ is constant on I and so $h'(x) = 0$ for all $x \in (a, b)$. So by Corollary 1, $h(x) = k$ in I. Therefore $f(x) - g(x) = k$ and $f(x) = g(x) + k$. *QED*

Example. Page 283 number 34.

Theorem. The following **Properties of Logarithms** are stated on page 52. We now use calculus to justify these properties. For any numbers $a > 0$ and $x > 0$ we have

1. $\ln ax = \ln a + \ln x$
2. $\ln \frac{a}{x} = \ln a - \ln x$
3. $\ln \frac{1}{x} = -\ln x$
4. $\ln x^r = r \ln x$ where r is rational.

Proof. First for **1**. Notice that

$$\frac{d}{dx}[\ln ax] = \frac{1}{ax}\frac{d}{dx}[ax] = \frac{1}{ax}a = \frac{1}{x}.$$

This is the same as the derivative of $\ln x$. Therefore by Corollary 2 to the Mean Value Theorem, $\ln ax$ and $\ln x$ differ by a constant, say $\ln ax = \ln x + k_1$ for some constant k_1. By setting $x = 1$ we need $\ln a = \ln 1 + k_1 = 0 + k_1 = k_1$. Therefore $k_1 = \ln a$ and we have the identity $\ln ax = \ln a + \ln x$.

Now for **2**. We know by **1**:

$$\ln \frac{1}{x} + \ln x = \ln\left(\frac{1}{x}x\right) = \ln 1 = 0.$$

Therefore $\ln \frac{1}{x} = -\ln x$. Again by **1** we have

$$\ln \frac{a}{x} = \ln \left(a \frac{1}{x} \right) = \ln a + \ln \frac{1}{x} = \ln a - \ln x.$$

Finally for **4**. We have by the Chain Rule (in the form of the previous theorem):

$$\frac{d}{dx}[\ln x^n] = \frac{1}{x^n}\frac{d}{dx}[x^n] = \frac{1}{x^n} n x^{n-1} = n\frac{1}{x} = n\frac{d}{dx}[\ln x] = \frac{d}{dx}[n \ln x].$$

As in the proof of **1**, since $\ln x^n$ and $n \ln x$ have the same derivative, we have $\ln x^n = n \ln x + k_2$ for some k_2. With $x = 1$ we see that $k_2 = 0$ and we have $\ln x^n = n \ln x$. *Q.E.D.*

Theorem. For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$.
2. $e^{-x} = \frac{1}{e^x}$
3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Note. The proofs are based on the definition of $y = e^x$ in terms of $x = \ln y$ and properties of the natural logarithm function.