Chapter 5. Integration

5.2 Sigma Notation and Limits of Finite Sums

Note. We use the *sigma notation* to denote sums:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

Examples. Page 369 number 2, page 370 number 18.

Note. We can verify (by mathematical induction):

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Examples. Page 370 number 24 and 28.

Definition. A *partition* of the interval $[a,b]$ is a set

$$P = \{x_0, x_1, \ldots, x_n\} \text{ where } a = x_0 < x_1 < \cdots < x_n = b.$$

partition P determines n closed *subintervals*

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

The length of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$.

Note. We now estimate the area bounded between a function $y = f(x)$ and the x-axis. We make the convention that the area bounded **above** the x-axis and below the function is **positive**, and the area bounded **below** the x-axis and above the curve is **negative**. We estimate this "area" by choosing a $c_k \in [x_{k-1}, x_k]$ and we use $f(c_k)$ as the "height" of a rectangle with base $[x_{k-1}, x_k]$. Then a partition P of $[a, b]$ can be used to estimate this "area" by adding up the "area" of these rectangles.

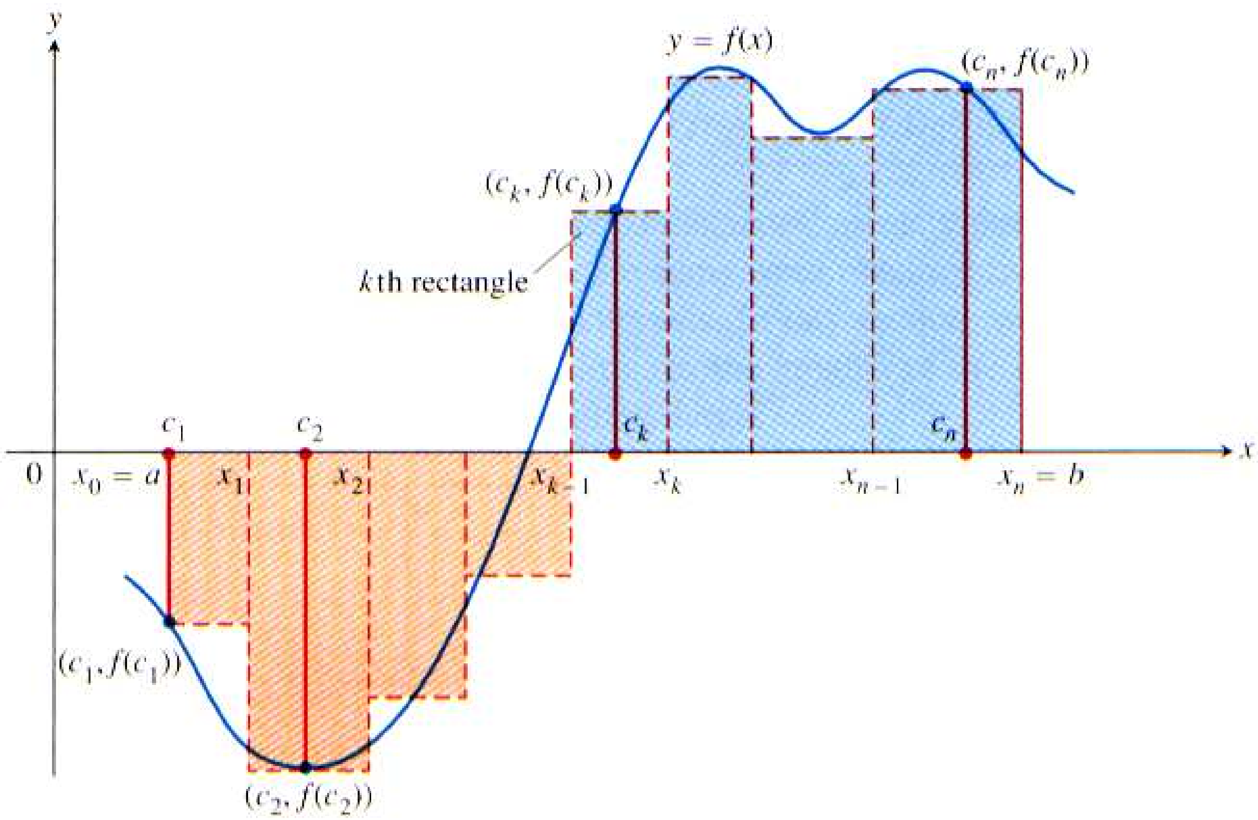

Figure 5.9, page 368

Definition. With the above notation, a *Riemann sum of f on the interval* $[a,b]$ is a sum of the form

$$s_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k.$$

Example. Page 370 number 30.

Definition. The *norm* of a partition $P = \{x_0, x_1, \ldots, x_n\}$ of interval $[a,b]$, denoted $\|P\|$, is largest subinterval:

$$\|P\| = \max_{1 \le k \le n} \Delta x_k = \max_{1 \le k \le n} (x_k - x_{k-1}).$$

Note. If $\|P\|$ is "small," then a Riemann sum is a "good" approximation of the "area" described above.