Chapter 5. Integration

5.3 The Definite Integral

Definition. Let f be a function defined on a closed interval $[a, b]$. We say that a number I is the *definite integral of* f *over* $[a, b]$ and that I is the limit of the Riemann sums if the following condition is satisfied: Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of $c_k \in [x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n} f(c_k)\,\Delta x_k - I\right| < \epsilon.$$

We denote $I = \int_a^b f(x)\,dx$ and say that f is *integrable* on $[a, b]$.

Note. We can also say:

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx.$$

Example. Page 379 number 6.

Note. When we deal with applications of integration, we will often think of definite integrals as sums. Notice, however, that strictly speaking they are not sums, but they are *limits* of sums.

Note. We have now introduced three ideas, each different from the other, but each related to the other (as we will see when we state the Fundamental Theorem of Calculus). We have:

Name of Object	**Type of Object**
ANTIDERIVATIVE	*FUNCTION*
INDEFINITE INTEGRAL	*COLLECTION* or *SET*
DEFINITE INTEGRAL	*NUMBER*

Antiderivatives and indefinite integrals are related by the fact that the indefinite integral of a function f is the set of all antiderivatives of f. The Fundamental Theorem of Calculus, to be seen in the next section, will relate antiderivatives and definite integrals (and therefore will relate definite and indefinite integrals).

Theorem 1. The existence of Definite Integrals. All continuous functions are integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

Theorem 2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable.

1. *Order of Integration*: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (this in fact is a definition)

2. *Zero*: $\int_a^a f(x)\,dx = 0$ (this too is a definition)

3. *Constant Multiple*: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

4. *Sum and Difference*: $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity*: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality*: If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \leq \int_a^b f(x)\,dx \leq \max f \cdot (b-a).$$

7. *Domination*: $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Example. Page 380 number 10.

Note. If we partition $[a, b]$ into n pieces of equal length $(b - a)/n$, then the partition if *regular*. We can then evaluate the limit $\|P\| \to 0$ by letting $n \to \infty$. This can be used to evaluate definite integrals. If we do so, then these formulas are useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Example. Use a regular partition of $[0, 1]$ with $c_k = x_k$ to evaluate $\int_0^1 x^2\,dx$. Notice that in page 377 Example 4 it is shown that $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$, and in page 381 number 76 it is shown that $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$.

Example. Page 380 number 34.

Definition. If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the *area under the curve* $y = f(x)$ *from* a *to* b is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

Example. Page 380 number 18.

Definition. If f is integrable on $[a,b]$, then its *average (mean) value* of $[a,b]$ is

$$\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example. Page 381 number 62.

Example. Page 381 number 63. There's an error in this one!