Chapter 5. Integration

5.4 The Fundamental Theorem of Calculus

Theorem 3. The Mean Value Theorem for Definite Integrals.

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

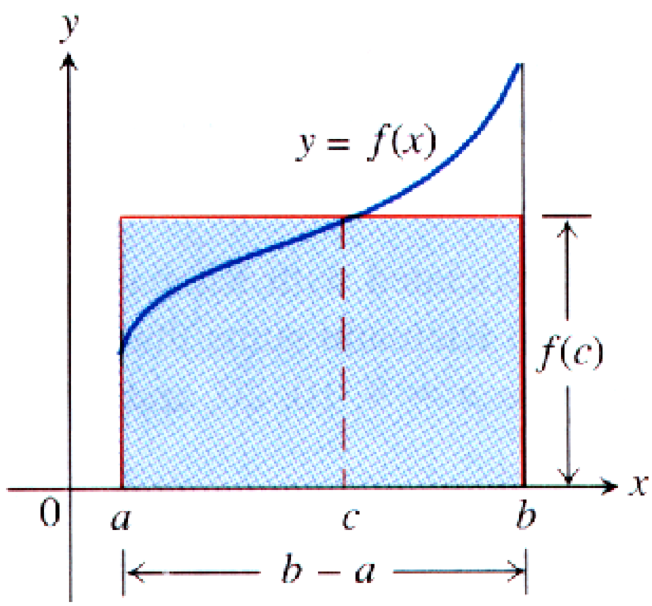

Figure 5.16, page 383

Proof of Theorem 3. By the Max-Min Inequality from Section 5.3, we have

$$\min f \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \max f.$$

Since f is continuous, f must assume any value between $\min f$ and $\max f$, including $\dfrac{1}{b-a}\int_a^b f(x)\,dx$ by the Intermediate Value Theorem. *Q.E.D.*

Theorem 4. The Fundamental Theorem of Calculus, Part 1. If f is continuous on $[a, b]$ then the function

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point x in $[a, b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

Proof. Notice that

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

So

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}[F(x+h) - F(x)] = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Since f is continuous, Theorem 2 implies that for some $c \in [x, x+h]$ we have

$$f(c) = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Since $c \in [x, x+h]$, then $\lim_{h\to 0} f(c) = f(x)$ (since f is continuous at x). Therefore

$$\begin{aligned} \frac{dF}{dx} &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \\ &= \lim_{h\to 0} f(c) = f(x) \end{aligned}$$

Q.E.D.

Example. Page 392 numbers 42 and 44.

Theorem 4. The Fundamental Theorem of Calculus, Part 2. If f is continuous at every point of $[a, b]$ and if F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. We know from the first part of the Fundamental Theorem (Theorem 3a) that

$$G(x) = \int_a^x f(t)\,dt$$

defines *an* antiderivative of f. Therefore if F is *any* antiderivative of f, then $F(x) = G(x) + k$ for some constant k. Therefore

$$\begin{aligned} F(b) - F(a) &= [G(b) + k] - [G(a) + k] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 \\ &= \int_a^b f(t)\,dt. \end{aligned}$$

QED

Examples. Page 392 numbers 16 and 60.

Example. Find the linearization of $g(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt$ at $a = -1$.