Chapter 6. Applications of Definite Integrals

6.4 Moments and Centers of Mass

Definition. Suppose masses $m_1, m_2, \ldots, m_n$ are distributed along the x-axis at coordinates $x_1, x_, \ldots, x_m$ respectively. The *moment of the system about the origin* is

$$m_1x_1 + m_2x_2 + \cdots + m_nx_n.$$

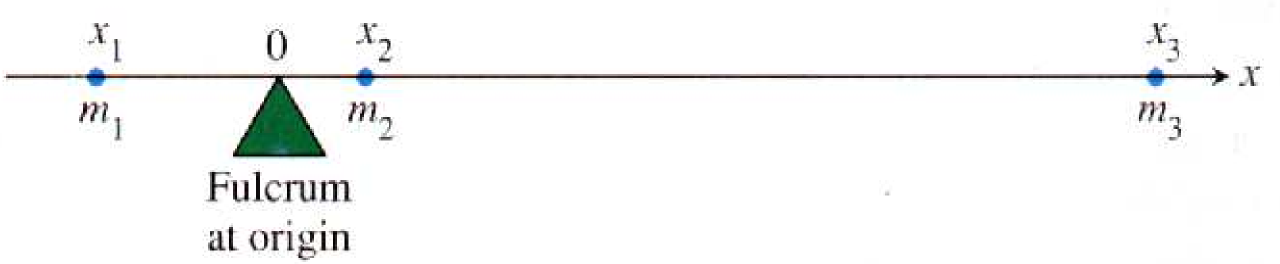

Figure from page 454.

Note. If g is the gravitational constant, then the *torque* of the above system is

$$g(m_1x_1 + m_2x_2 + \cdots + m_nx_n).$$

If we try to balance the system at the origin then:

1. It tips down on the left side if torque is negative.
2. It tips down on the right side if torque is positive.
3. It is balanced if torque is zero.

Definition. The *moment* of the system above about the point $\overline{x}$ is

$$(x_1 - \overline{x})m_1 + (x_2 - \overline{x})m_2 + \cdots + (x_n - \overline{x})m_n = \sum_{k=1}^{n}(x_k - \overline{x})m_k.$$

The *torque* about $\overline{x}$ is moment times the gravitational constant (and so is measured in units of force times distance). The *center of mass* is the coordinate $\overline{x}$ about which the moment is 0:

$$\overline{x} = \frac{\sum_{k=1}^{n} m_kx_k}{\sum_{k=1}^{n} m_k}.$$

Note. The center of mass is the moment about the origin divided by total mass.

Definition. A thin straight wire whose density is given by $\delta(x)$ has the following:

Moment about the Origin: $M_0 = \int_a^b x\delta(x)\,dx$

Mass: $M = \int_a^b \delta(x)\,dx$

Center of Mass: $\overline{x} = \dfrac{M_0}{M}$.

Example. Page 463 number 12.

Definition. Suppose masses $m_1, m_2, \ldots, m_k$ are placed in the (x, y)-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ respectively. Then define

1. Mass: $\sum_{k=1}^{n} m_k$
2. Moment About the x-Axis: $M_x = \sum_{k=1}^{n} m_k y_k$
3. Moment About the y-Axis: $M_y = \sum_{k=1}^{n} m_k x_k$.

The *center of mass* is then $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{M_y}{M}$ and $\overline{y} = \dfrac{M_x}{M}$.

Note. The above definition is really just a two dimensional version of our original one dimensional "moment about the origin" definition.

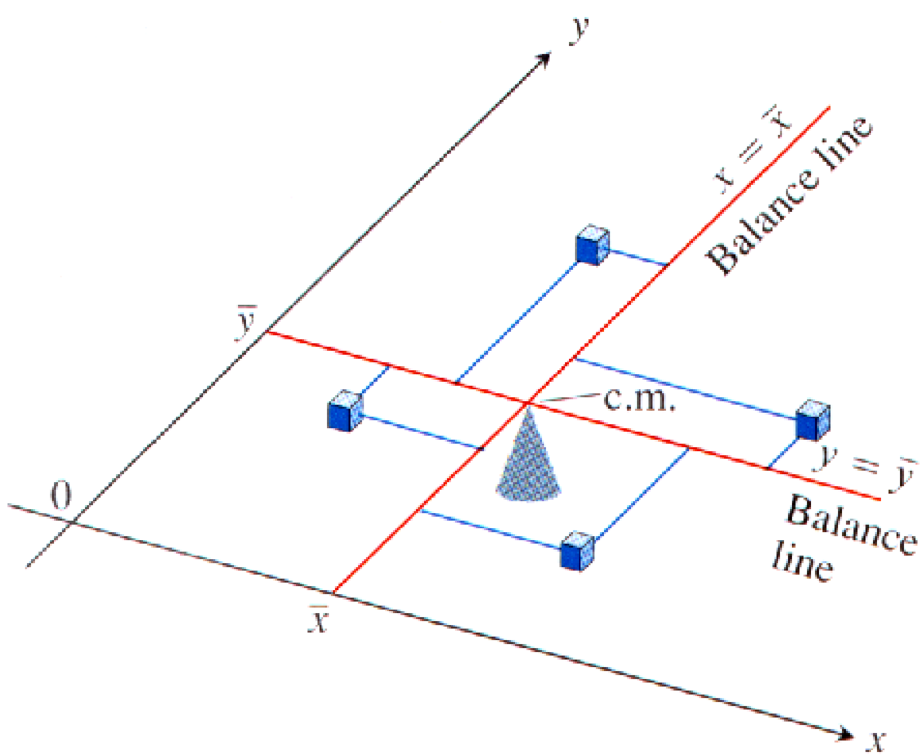

Figure 6.33 page 458

Note. Consider a region in the plane:

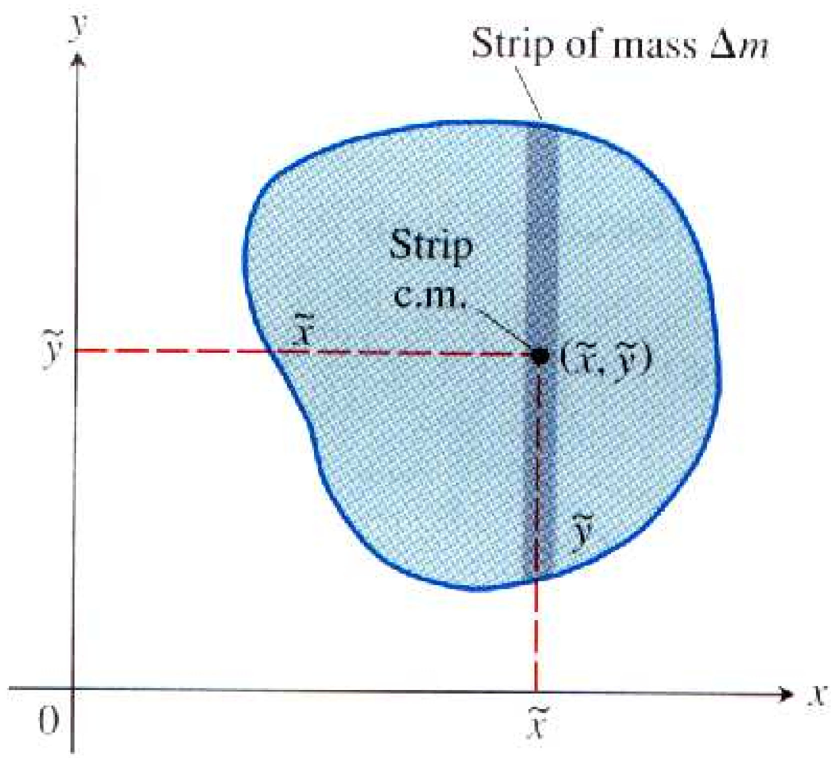

Figure 6.34 page 458

Take a slice (dx or dy) of mass dm. Suppose the center of mass of this slice is $\tilde{x}, \tilde{y}$).

Definition. For the region above, define

1. Moment About the x-Axis: $M_x = \int \tilde{y}\, dm$.

2. Moment About the y-Axis: $M_y = \int \tilde{x}\, dm$.

3. Mass: $\int dm$.

The *center of mass* is then $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{My}{M}$ and $\overline{y} = \dfrac{M_x}{M}$.

Examples. Page 463 number 22a (answer: $\overline{x} = 4/\pi, \overline{y} = 4/\pi$). Page 463 number 28 ($dm = \delta(x)\, dx$). Page 464 number 40 ($dm =$ arc length $\times \delta$).

HAVE A NICE DAY!