Chapter 10. Infinite Sequences and Series

10.1 Sequences

Note. We now shift gears and consider a more theoretical, less tangible concept than those with which we have recently dealt.

Definition. An infinite *sequence* of numbers is a function whose domain is the set of positive integers. We denote the sequence $\{f(n) \mid n \in \mathbb{N}\}$ as $\{a_n\}$ where $f(n) = a_n$.

Definition. The sequence $\{a_n\}$ *converges* to the number L if for every $\epsilon > 0$ there exists an integer N such that for all $n > N$ we have

$$|a_n - L| < \epsilon.$$

If no such number L exists, then the sequence $\{a_n\}$ *diverges*. If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$ and call L the *limit* of the sequence.

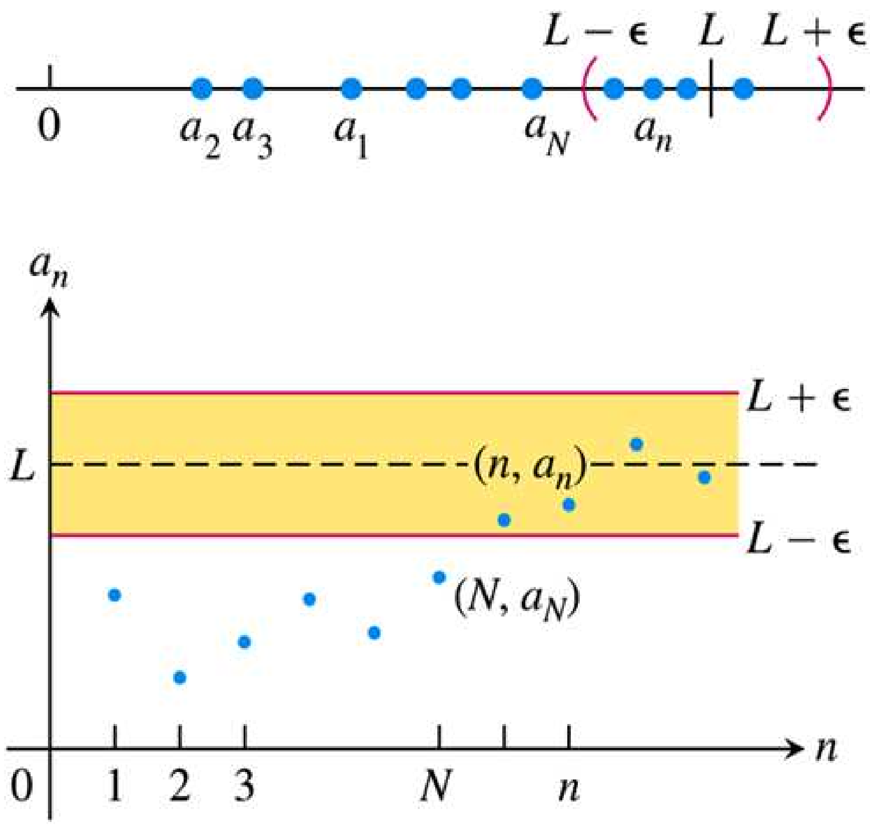

Figure 10.2 page 552

Example 1a. Let $a_n = \dfrac{1}{n}$. Prove that $\lim\limits_{n\to\infty} a_n = 0$.

Proof. Let $\epsilon > 0$ be given. Let N be an integer greater than $1/\epsilon$. Then for all $n > N$ we have $0 < a_n = 1/n < 1/N < \epsilon$, or $|a_n - 0| < \epsilon$. Therefore $\lim\limits_{n\to\infty} a_n = 0$. *Q.E.D.*

Definition. The sequence $\{a_n\}$ *diverges to infinity* if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds, we write

$$\lim_{n\to\infty} a_n = \infty \text{ or } a_n \to \infty.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ *diverges to negative infinity* and write

$$\lim_{n\to\infty} a_n = -\infty \text{ or } a_n \to -\infty.$$

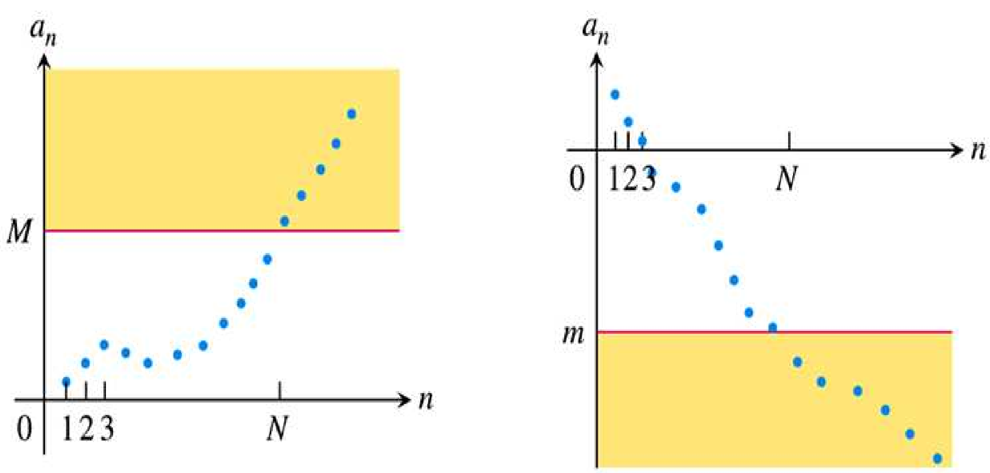

Figure 10.3 page 553

Example. Page 559 Number 30.

Theorem 1. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ then

1. *Sum Rule*: $\lim_{n\to\infty}(a_n + b_n) = A + B$.

2. *Difference Rule*: $\lim_{n\to\infty}(a_n - b_n) = A - B$.

3. *Product Rule*: $\lim_{n\to\infty}(a_n b_n) = AB$.

4. *Constant Multiple Rule*: $\lim_{n\to\infty}(kb_n) = kB$.

5. *Quotient Rule*: $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{A}{B}$, if $B \neq 0$.

Note. The proofs for each of these is similar to the proofs of the corresponding results for functions.

Example. Page 559 Number 32.

Theorem 2. The Sandwich Theorem for Sequences.

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N and is $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Example. Page 559 Number 46.

Theorem 3. The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Theorem 4. Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then $\lim_{x\to\infty} f(x) = L$ implies that $\lim_{n\to\infty} a_n = L$.

Note. Theorem 4 allows us to use L'Hôpital's Rule on sequences.

Example. Page 555 Example 7.

Theorem 5. The following six sequences converge to the limits listed below:

1. $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0.$

2. $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1.$

3. $\displaystyle\lim_{n\to\infty} x^{1/n} = 1$ for $x > 0$.

4. $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$.

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$.

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

Definition. A sequence can be defined recursively by giving:

1. The value(s) of the initial term or terms and

2. A rule, called a *recursion formula*, for calculating any later term from terms that precede it.

Example 10c. The *Fibonacci sequence* is defined recursively as: $a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$. The first few terms therefore are: 1, 1, 2, 3, 5, 8, 13, 21, 44, 65, ...

Definition. A sequence $\{a_n\}$ with the property that $a_n \leq a_{n+1}$ for all n is called a *nondecreasing sequence.* It is called *nonincreasing* if $a_n \geq a_{n+1}$ for all n. A sequence is *monotone* if it is either nondecreasing or nonincreasing.

Definition. A sequence $\{a_n\}$ is *bounded from above* if there exists a number M such that $a_n \leq M$ for all n. The number M is an *upper bound* for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the *least upper bound* for $\{a_n\}$. The sequence is *bounded from below* if there exists a number m such that $m \leq a_n$ for all n. The number m is a *lower bound* for $\{a_n\}$. If it is bounded from above and below, then $\{a_n\}$ is a *bounded sequence*.

Theorem 6. The Monotonic Sequence Theorem.
If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

Note. The proof of Theorem 6 depends heavily on the very definition of the real numbers. Appendix A.6 gives some details in the form of the *Completeness Property*. This property states that every *set* of real numbers with an upper bound, has a least upper bound.

Example. Page 560 number 86; page 561 Number 131.