Chapter 10. Infinite Sequences and Series

10.10 The Binomial Series and Applications of Taylor Series

Note. If we define $f(x) = (1+x)^m$, then we find that the Taylor series for f is

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define (for **any** m)

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}.$$

This is called the *binomial series.* It converges for $|x| < 1$.

Example. Page 620 Number 10.

Example. We can use series to evaluate definite and indefinite integrals. For example, consider Page 621 Number 26.

Note. We can take the binomial series and replace x with x^2 to find that

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Integrating we find that for $|x| \leq 1$ (see page 617 for details)

$$\tan^{-1} x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Plugging $x = 1$ into this formula, we find that we have a series representation for π: $\dfrac{\pi}{4} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. Notice that this is an alternating series, and hence we could approximate π by taking partial sums of this series and applying the Alternating Series Estimation Theorem. When you hear that people have calculated π to a million digits, or some such, then they are using a method similar to this approach.

Note. We can sometimes evaluate indeterminate forms by expressing the functions involved as Taylor series.

Example. Page 618 Example 6. Evaluate $\displaystyle\lim_{x \to 0} \frac{\sin x - \tan x}{x^3}$ using

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \tan x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)}.$$

Note. We can establish the following Maclaurin series (Table 10.1):

1. $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots x^n = \displaystyle\sum_{n=0}^{\infty} x^n \ \ (|x| < 1)$

2. $\dfrac{1}{1+x} = 1 - x + x^2 + \cdots + (-x)^n + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n \ \ (|x| < 1)$

3. $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

4. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ (all x)

5. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots + (-1)^n \dfrac{x^n}{(2n)!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ (all x)

6. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots - (-1)^{n-1} \dfrac{x^n}{n} + \cdots \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $(-1 < x \leq 1)$

7. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots + (-1)^n \dfrac{x^{2n+1}}{2n+1}$ $(|x| \leq 1)$

Example. Page 621 number 48; page 621 number 56.

Example. Find a series for $f(x) = e^{-x^2}$. Use the series to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ to the nearest 0.01.

Note. On pages 619 and 620, the text uses series to argue that for all real numbers θ we have $e^{i\theta} = \cos\theta + i\sin\theta$. This is called *Euler's identity* and when $\theta = \pi$ we have the common t-shirt logo $e^{i\pi} = -1$.