Chapter 10. Infinite Sequences and Series

10.2 Infinite Series

Definition. Given a sequence of numbers $\{a_n\}$, an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an *infinite series*. The number a_n is the n^{th} *term* of the series. The *partial sums* of the series are the elements of the sequence

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

If the sequence of partial sums has a limit L, then we say that the *series converges* to the sum L and write $\sum_{n=1}^{\infty} a_n = L$. If the sequence of partial sums of the series does not converge, we say that the series *diverges*.

Definition. A *geometric series* is a series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The parameter r is called the *ratio* of the series.

Theorem. The geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

converges to the sum $a/(1-r)$ if $|r| < 1$ and diverges if $|r| \geq 1$.

Example. Page 569 Number 2. Notice also Example 4 on page 565.

Example. Example 5 page 565. Consider the partial sums of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ and find the sums of the series.

Solution. We can apply the partial fractions idea to find that $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. Then for the nth partial sum, we find that

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k \cdot (k+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k}\right) + \left(\frac{1}{k} - \frac{1}{k+1}\right) \end{aligned}$$

$$= \frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \cdots + \left(-\frac{1}{k} + \frac{1}{k}\right) - \frac{1}{k+1}$$
$$= 1 - \frac{1}{k+1}.$$

Since $s_k \to 1$, then the series sums to 1.

Example. Does the series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ converge?

Theorem 7. Test for Divergence
If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$. Equivalently, if $\displaystyle\lim_{n\to\infty} a_n$ does not exist or is not 0, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Note. Notice that Theorem 7 is a test for **divergence**! If $\displaystyle\lim_{n\to\infty} a_n = 0$ then it **does not say** that the series converges. As we will see, there are series for which the terms approach 0, but the series still diverges.

Example. Page 569 Number 28.

Theorem 8. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent series, then

1. *Sum Rule*: $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B$.

2. *Difference Rule*: $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n = A - B$.

3. *Constant Multiple Rule*: $\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n = kA$ for any number k.

Proof. The proof of each follows in a way similar to the proof of Theorem 1 of section 10.1 for sequences. See page 567. *Q.E.D.*

Example. Page 569 Number 12.

Note. Similar to Theorem 7 for convergent series, we have the following for divergent series:

Theorem. Every nonzero constant multiple of a divergent series diverges. If $\sum_{=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n+b_n)$ and $\sum_{n=1}^{\infty}(a_n-b_n)$ both diverge.

Example. Page 570 numbers 88 and 90.