Chapter 10. Infinite Sequences and Series

10.3 The Integral Test

Note. Given a series $\displaystyle\sum_{n=1}^{\infty} a_n$ we have two questions:

1. Does the series converge?

2. If it converges, what is its sum?

Corollary of Theorem 6. A series $\displaystyle\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if its partial sums are bounded from above.

Proof. Theorem 6 (of section 10.1) implies that a monotonic increasing sequence which is bounded above must converge. A positive term series will have partial sums which form a monotonic increasing sequence. Since we have hypothesized that the sequence of partial sums is bounded, the result follows. *Q.E.D.*

Theorem 9. The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\displaystyle\sum_{n=N}^{\infty} a_n$ and the integral $\displaystyle\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Proof. Since a finite number of terms does not affect the convergence of a series, we may assume that $N = 1$ without loss of generality. Under the hypotheses of f as continuous and decreasing, we can consider the following rectangles (left):

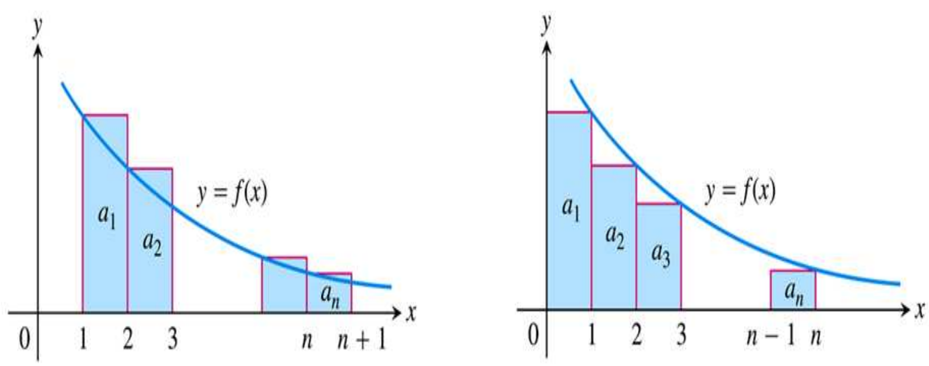

Figure 10.11 page 572

The areas of the rectangles are $a_1, a_2, a_3, \ldots, a_n$, and since f is decreasing, these rectangles are *circumscribed* over f and we have

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n.$$

If we consider *inscribed* rectangles, then we have the picture above (right). Excluding a_1, we see that

$$a_2 + a_3 + a_4 + \cdots + a_n \leq \int_1^n f(x)\,dx,$$

or that

$$a_1 + a_2 + a_3 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx.$$

Therefore we know that

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + a_3 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,x.$$

If $\displaystyle\int_1^\infty f(x)\,dx$ is finite, then the right-hand inequality shows that $\displaystyle\sum_{n=1}^\infty a_n$ is finite. If $\displaystyle\int_1^\infty f(x)\,dx$ is infinite, then the left-hand inequality shows that $\displaystyle\sum_{n=1}^\infty a_n$ is infinite. *Q.E.D.*

Example. Page 575 Number 4.

Theorem. ***p*-Series**

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

is called a *p-series*. A p series converges if $p > 1$ and diverges if $p \leq 1$.

Proof. We prove this using the Integral Test. First, suppose $p \neq 1$. Then

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty} \left(\int_1^b \frac{dx}{x^p} \right) = \lim_{b\to\infty} \left(\frac{x^{-p+1}}{-p+1} \right) \Bigg|_1^b$$

$$= \lim_{b\to\infty} \left(\frac{1}{1-p} (b^{-p+1} - 1) \right) = \lim_{b\to\infty} \frac{1}{1-p} \left(\frac{1}{b^{p-1}} - 1 \right)$$

$$= \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore both the integral and the series converge if $p > 1$, and both diverge if $p < 1$. Next, suppose that $p = 1$. Then

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \left(\ln x \,\Big|_1^b \right) = \lim_{b\to\infty} (\ln b - \ln 1) = \infty.$$

By the integral test, the series diverges when $p = 1$. *Q.E.D.*

Definition. The p-series with $p = 1$ is the *harmonic series*.

Note. Let's briefly explore the *rate* at which the harmonic series diverges. How many terms must we add in the harmonic series to get a partial sum greater than 20. Consider these two graphs:

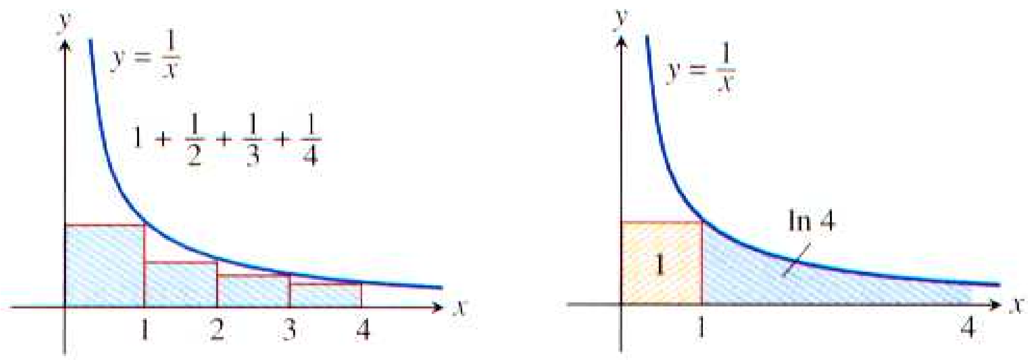

Figure 8.14 from Edition 10

We see that the 4th partial sum is less than $1 + \ln 4$, and in general the n^{th} partial sum will be less than $1 + \ln n$. Therefore we need *at least* $1 + \ln n > 20$, or $n > e^{19} \approx 178,482,301$. We can use a similar argument with circumscribed rectangles to see that the n^{th} partial sum is greater than $\ln(n+1)$, and so we find that to get the partial sum greater than 20, we would need *at most* $n = e^{20} - 1 \approx 485,165,194$.

Theorem. Bounds for the Remainder in the Integral Test.

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and that $\sum a_n$ converges to S. Then the remainder $r_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Proof. Notice that $R_n = S - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$. Consider Figure 10.11a again. By considering the areas of the rectangles with the area under the curve $y = f(x)$ for $x \geq n$, we see that

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^{\infty} f(x)\,dx.$$

Similarly, From Figure 10.11b, we find an upper bound with

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_{n}^{\infty} f(x)\,dx.$$

Q.E.D.

Example. Page 575 Number 32, Page 576 Number 50.