Chapter 10. Infinite Sequences and Series

10.4 Comparison Tests

Theorem 10. The (Direct) Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with no negative terms.

(a) $\sum_{n=1}^{\infty} a_n$ converges if there is a convergent series $\sum_{n=1}^{\infty} c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N.

(b) $\sum_{n=1}^{\infty} a_n$ diverges if there is a divergent series of nonnegative terms $\sum_{n=1}^{\infty} d_n$ with $a_n \geq d_n$ for all $n > N$, for some integer N.

Proof. For part **(a)**, the partial sums of $\sum_{n=1}^{\infty} a_n$ are bounded above by

$$M = a_1 + a_2 + \cdots + a_n + \sum_{n=N+1}^{\infty} c_n.$$

Therefore by the corollary to Theorem 6, the result holds.

For part **(b)**, the partial sums of $\sum_{n=1}^{\infty} a_n$ are not bounded above (for if they were, then the partial sums of $\sum_{n=1}^{\infty} d_n$ would be bounded and it would be convergent). Therefore $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges. *Q.E.D.*

Example. Page 580 Number 20.

Theorem 11. Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N a positive integer).

1. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

2. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

3. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Proof of (1). Since $c/2 > 0$, there exists an integer N such that for all $n > N$ we have $\left|\dfrac{a_n}{b_n} - c\right| < \dfrac{c}{2} \equiv \epsilon$. So for $n > N$ it follows that

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2},$$

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2},$$

$$\left(\frac{c}{2}\right) b_n < a_n < \left(\frac{3c}{2}\right) b_n.$$

If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges then $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3c}{2}\right) b_n$ converges and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test. If $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty}\left(\frac{c}{2}\right) b_n$ diverges and $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges by the Direct Comparison Test. *Q.E.D.*

Example. Page 580 Numbers 28 and 38.