Chapter 10. Infinite Sequences and Series

10.5 The Ratio and Root Tests

Theorem 12. The Ratio Test
Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then

1. the series converges if $\rho < 1$,

2. the series diverges if $\rho > 1$ or if ρ is infinite, and

3. the test is inconclusive if $\rho = 1$ (that is, the series could diverge or converge — the Ratio Test tells us nothing).

Proof. (a) Let r be a number between ρ and 1. Then the number $\epsilon = r - \rho$ is positive. Since $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, then there exists positive integer N such that for all $n \geq N$ we have a_{n+1}/a_n within ϵ of ρ. In particular, for $n \geq N$ we have $\dfrac{a_{n+1}}{a_n} < \rho + \epsilon = r$. That is,

$$\begin{aligned} a_{N+1} &< ra_N \\ a_{N+2} &< ra_{N+1} < r^2 a_N \end{aligned}$$

$$
\begin{aligned}
a_{N+3} &< ra_{N+2} < r^3 a_N \\
&\vdots \\
a_{N+m} &< ra_{N+m-1} < r^m a_N.
\end{aligned}
$$

If we define the series

$$\sum_{n=1}^{\infty} c_n \equiv a_1 + a_2 + \cdots + a_{N-1} + a_N(1 + r + r^2 + \cdots),$$

then we see that this new series converges, for it is (eventually) a geometric series with ratio r between 0 and 1. Therefore by the Direct Comparison Test, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(b) With $1 < \rho \leq \infty$, we must eventually have (that is, for all $n \geq M$ where M is some positive integer) $\dfrac{a_{n+1}}{a_n} > 1$. That is, $0 < a_M < a_{M+1} < a_{M+2} < \cdots$. Therefore the sequence $\{a_n\}$ either diverges or has a limit greater than 0. So by the Test for Divergence, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

(c) Consider the two series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. For both series, $\rho = 1$, but the first series diverges, while the second series converges. *Q.E.D.*

Note. The Ratio Test (if applicable) is easier to use than the Direct Comparison Test. This is because you don't need to find a second series which has the appropriate behavior (in terms of convergence or divergence) and satisfies the appropriate inequalities. The Ratio Test is particularly useful when the series involves factorials.

Example. Page 585 numbers 4 and 34.

Theorem 13. The Root Test
Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geq 0$ for $n \geq N$ and suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$. Then

(a) the series converges if $\rho < 1$,

(b) the series diverges if $\rho > 1$ or ρ is infinite, and

(c) the test is inconclusive if $\rho = 1$.

Note. Again, the Root Test doesn't require a second series and is easier to use than the Direct Comparison Test.

Example. Page 585 numbers 16 and 56.