Chapter 10. Infinite Sequences and Series

10.6 Alternating Series, Absolute and Conditional Convergence

Note. The convergence tests investigated so far apply only to series with nonnegative terms. In this section, we learn how to deal with series that may have negative terms. An important example is the alternating series, whose terms alternate in sign. We also learn which convergent series can have their terms rearranged (that is, changing the order in which they appear) without changing their sum.

Definition. A series in which terms are alternately positive and negative is an *alternating series*.

Theorem 14. The Alternating Series Test (Leibniz's Theorem)

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following are satisfied:

1. The u_n's are all positive,

2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N, and

3. $\lim\limits_{n\to\infty} u_n = 0$.

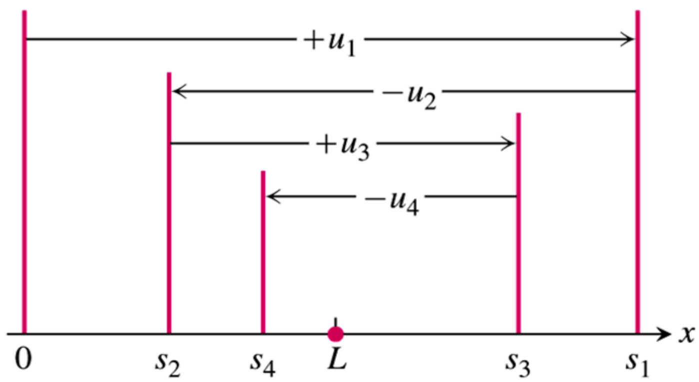
Proof. If n is an even integer, say $n = 2m$, then the sum of the first n terms is

$$\begin{aligned} s_{2m} &= (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}) \\ &= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}. \end{aligned}$$

From the first equality, we see that s_{2m} is the sum of m nonnegative terms, since each term in parentheses is positive or zero. Hence $s_{2m+2} \geq s_{2m}$, and the sequence $\{s_{2m}\}$ is nondecreasing. The second equality implies that $s_{2m} \leq u_1$. Since $\{s_{2m}\}$ is nondecreasing and bounded from above by u_1, it has a limit L.

If n is an odd integer, say $n = 2m + 1$, then the sum of the first n terms is $s_{2m+1} = s_{2m} + u_{2m+1}$. Since $\lim\limits_{n\to\infty} u_n = 0$, then $\lim\limits_{m\to\infty} s_{2m+1} = \lim\limits_{m\to\infty} s_{2m} + u_{2m+1} = L + 0 = L$. Combining these results, we see that $\lim\limits_{n\to\infty} s_n = L$ (see Exercise 131 section 10.1). *Q.E.D.*

Note. The following figure illustrates how Theorem 14 works:



Page 586 Figure 10.13

This figure shows how the alternating series converges. The partial sums keep "overshooting" the limit as they go back and forth on the number line, gradually closing in as the terms tend to zero. If we stop at the n^{th} partial sum, we know that the next term $(\pm u_{n+1})$ will again cause us to overshoot the limit in the positive direction or negative direction, depending on the sign carried by u_{n+1}. This gives us a convenient bound for the *truncation error*, which we state in the following theorem.

Theorem 15. The Alternating Series Estimation Theorem
If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the conditions of Theorem 8, then the truncation error for the n^{th} partial sum is less than u_{n+1} and has the same sign as the unused term.

Example. Prove that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent, but that the corresponding series of absolute values is not convergent. Find a bound for the truncation error after 99 terms.

Definition. A series $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges absolutely* if the corresponding series of absolute values $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges. A series that converges but does not converge absolutely *converges conditionally*.

Examples. Page 591 Numbers 8 and 18.

Theorem 10. The Absolute Convergence Test
If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Proof. For each n,

$$-|a_n| \leq a_n \leq |a_n|, \text{ so } 0 \leq a_n + |a_n| \leq 2|a_n|.$$

If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then $\displaystyle\sum_{n=1}^{\infty} 2|a_n|$ converges and, by the (Direct) Comparison Test, the nonnegative series $\displaystyle\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges. The equality

$a_n = (a_n + |a_n|) - |a_n|$ lets us express $\sum_{n=1}^{\infty} a_n$ as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

Therefore $\sum_{n=1}^{\infty} a_n$ converges. *Q.E.D.*

Example. Page 591 Number 32.

Example. Page 589 Example 5 By Theorem 14, the alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges for all $p > 0$. We have seen that for $p > 1$ the series, in fact, converges absolutely. However, since for $0 < p \leq 1$ the regular p-series diverge, we see that the alternating p-series are conditionally convergent for these values of p.

Theorem 17. The Rearrangement Theorem for Absolutely Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely and $b_1, b_2, \ldots, b_n, \ldots$, is any arrangement of the sequence $\{a_n\}$, then $\sum_{n=1}^{\infty} b_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

Note. A more interesting result than the Rearrangement Theorem is the following:

Theorem. A conditionally convergent series can be rearranged to converge to any desired limit (including $-\infty$ or $+\infty$), or to diverge.

Example. We can rearrange the alternating harmonic series to converge to 1. We start with the first term $1/1$ and then subtract $1/2$. Next we add $1/3$ and $1/5$, which brings the total back to 1 or above. Then we add consecutive negative terms until the total is less than 1. We continue in this manner: When the sum is less than 1, add positive terms until the total is 1 or more, then subtract (add negative) terms until the total is again less than 1. This process can be continued indefinitely. Because both the odd numbered terms and the even-numbered terms of the original series approach 0 as $n \to \infty$, the amount by which our partial sums exceed 1 or fall below it approaches 0. So the new series converges to 1. The rearranged series starts like this:

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}+\frac{1}{5}-\frac{1}{4}+\frac{1}{7}+\frac{1}{9}-\frac{1}{6}+\frac{1}{11}+\frac{1}{13}-\frac{1}{8}+\frac{1}{15}+\frac{1}{17}-\frac{1}{10}+\frac{1}{19}$$
$$+\frac{1}{21}-\frac{1}{12}+\frac{1}{23}+\frac{1}{25}-\frac{1}{14}+\frac{1}{27}-\frac{1}{16}+\cdots$$