Chapter 10. Infinite Sequences and Series

10.7 Power Series

Definition. An expression of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

is a *power series about* $x = 0$. An expression of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

is a *power series about* $x = a$. The term $c_n(x-a)^n$ is the n^{th} *term*, the number a is the *center*.

Note. We are interested in finding the values of x for which the above power series converge. By convention, a power series centered at a always converges to c_0 for $x = a$ (notice that the summation notation implies that we consider 0^0, but this is just a short-coming of the notation - the 0^{th} term is always c_0).

Example. We know that $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$ forms a geometric series and converges for $|x| < 1$ to $\dfrac{1}{1-x}$.

Example. For what x does $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$ converge?

Solution. By the Ratio Test, the series of absolute values satisfies

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{3^{n+1}|x|^{n+1}}{(n+1)!} \frac{n!}{3^n|x|^n} = \lim_{n\to\infty} \frac{3|x|}{n+1} = 0.$$

Therefore this series converges for all x, and the original series converges absolutely for all x.

Corollary to Theorem 18. There are three possibilities for $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ with respect to convergence.

1. There is a positive number R such that the series diverges for $|x-a| > R$ but converges absolutely for $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.

2. The series converges for all x (that is, $R = \infty$).

3. The series converges at $x = a$ and diverges elsewhere (that is, $R = 0$).

Definition. The number R of the above corollary is the *radius of convergence*, and the set of all values of x for which the series converges is the *interval of convergence*.

Example. Page 595 Example 3b. Find the interval of convergence for $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$.

Note. To find the interval of convergence for a power series:

1. Use the Ratio Test (or the Root Test) to find the interval where the series converges absolutely.

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use a comparison Test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is $a - R < x < a + R$, then the series diverges for $|x - a| > R$.

Example. Page 600 Number 8.

Theorem 19. The Series Multiplication Theorem for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$ and if

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n.$$

Theorem 21. The Term-by-Term Differentiation Theorem

If $\sum_{n=1}^{\infty} c_n (x-a)^n$ converges for $a - R < x < a + R$ for some $R > 0$, it defines a function f:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \;\; a - R < x < a + R.$$

Such a function has derivatives of all orders inside the interval of convergence and is said to be *analytic*. We can obtain the derivatives by differentiating the original series term by term:

$$f'(x) \;=\; \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x) \;=\; \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

Note. The proof of Theorem 21 is found in an advanced calculus class or in an introductory analysis class (such as our MATH 4217/5217 Analysis). The following two theorems are other results from a more advanced class.

Theorem 22. The Term-by-Term Integration Theorem

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

converges for $a - R < x < a + R$ $(R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < A + R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

Example. Page 804 number 52.