Chapter 10. Infinite Sequences and Series

10.9 Convergence of Taylor Series

Note. We still have unanswered questions relevant to the generation of Taylor series from infinitely differentiable functions:

1. When does a Taylor series converge to its generating function?

2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

Theorem 23. Taylor's Theorem

If f and its first n derivatives f', f'', $\ldots$, f^n are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}+\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Theorem. Taylor's Formula

If f is differentiable through order $n+1$ in an open interval I containing a, then for each $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \text{ for some } c \text{ between } a \text{ and } x.$$

Note. If we can be insured that the remainder term R_n goes to 0 as $n \to \infty$, then the Taylor series will converge to the generating function. This is summarized in the following theorem.

Theorem 24. The Remainder Estimation Theorem.

If there are positive constants M and r such that $|f^{(n+1)}(t)| \le Mr^{n+1}$ for all t between a and x, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le M\frac{r^{n+1}|x-a|^{n+1}}{(n+1)!.}$$

If these conditions hold for every n and all the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Example. Page 613 numbers 2, 16, and 46.