Chapter 6. Applications of Definite Integrals

6.6. Moments and Centers of Mass

Definition. Suppose masses $m_1, m_2, \ldots, m_n$ are distributed along the x-axis at coordinates $x_1, x_, \ldots, x_m$ respectively. The *moment of the system about the origin* is

$$m_1x_1 + m_2x_2 + \cdots + m_nx_n.$$

Figure from page 402.

Note. If g is the gravitational constant, then the *torque* of the above system is

$$g(m_1x_1 + m_2x_2 + \cdots + m_nx_n).$$

If we try to balance the system at the origin then:

1. It tips down on the left side if torque is negative.
2. It tips down on the right side if torque is positive.
3. It is balanced if torque is zero.

Definition. The *moment* of the system above about the point $\overline{x}$ is

$$(x_1 - \overline{x})m_1 + (x_2 - \overline{x})m_2 + \cdots + (x_n - \overline{x})m_n = \sum_{k=1}^{n}(x_k - \overline{x})m_k.$$

The *torque* about $\overline{x}$ is moment times the gravitational constant (and so is measured in units of force times distance). The *center of mass* is the coordinate $\overline{x}$ about which the moment is 0:

$$\overline{x} = \frac{\sum_{k=1}^{n} m_kx_k}{\sum_{k=1}^{n} m_k}.$$

Note. The center of mass is the moment about the origin divided by total mass.

Figure from page 403.

Definition. A thin straight wire whose density is given by $\delta(x)$ has the following:

Moment about the Origin: $M_0 = \displaystyle\int_a^b x\delta(x)\,dx$

Mass: $M = \displaystyle\int_a^b \delta(x)\,dx$

Center of Mass: $\overline{x} = \dfrac{M_0}{M}$.

Example. A wire lies along the x-axis with density $\delta(x) = x^2$ kg/m, with x in meters. Find the center of mass of the wire.

Definition. Suppose masses $m_1, m_2, \ldots, m_k$ are placed in the (x, y)-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ respectively. Then define

1. Mass: $\displaystyle\sum_{k=1}^{n} m_k$
2. Moment About the x-Axis: $\displaystyle M_x = \sum_{k=1}^{n} m_k y_k$
3. Moment About the y-Axis: $\displaystyle M_y = \sum_{k=1}^{n} m_k x_k$.

The *center of mass* is then $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{M_y}{M}$ and $\overline{y} = \dfrac{M_x}{M}$.

Note. The above definition is really just a two dimensional version of our original one dimensional "moment about the origin" definition.

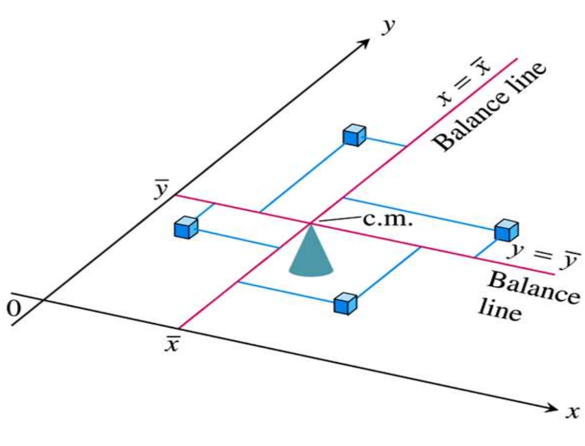

Figure 6.46 page 404.

Note. Consider a region in the plane:

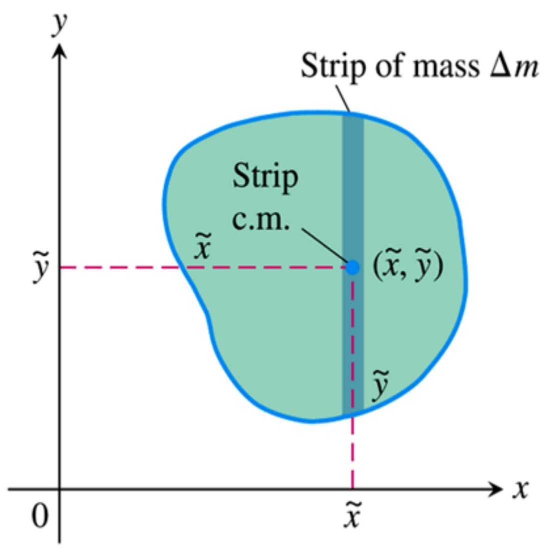

Figure 6.47 page 404

Take a slice (dx or dy) of mass dm. Suppose the center of mass of this slice is $\tilde{x}, \tilde{y}$).

Definition. For the region above, define

1. Moment About the x-Axis: $M_x = \displaystyle\int \tilde{y}\, dm$.
2. Moment About the y-Axis: $M_y = \displaystyle\int \tilde{x}\, dm$.
3. Mass: $\displaystyle\int dm$.

The *center of mass* is then $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{My}{M}$ and $\overline{y} = \dfrac{M_x}{M}$.

Examples. Page 411 number 8; page 412 number 16.

Theorem 1. Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then $V = 2\pi\rho A$.

Example. Page 410 Example 6.

Theorem 2. Pappus's Theorem for Surface Areas

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then $S = 2\pi\rho L$.

Example. Page 413 number 40.