Chapter 7. Integrals and Transcendental Functions

7.1. The Logarithm Defined as an Integral

Note. In this section, we introduce the natural logarithm function using definite integrals. However, to justify using the *logarithm* terminology we must show that the function we introduce satisfies the usual properties of logarithms. We will do so using our definition and the "calculus properties" which it satisfies.

Definition. For $x > 0$, define the *natural logarithm* function as

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

Note. It follows from the definition that for $x \geq 1$, $\ln x$ is the area under the curve $y = 1/t$ for $t \in [1, x]$. All we can currently tell from the definition, is that $\ln x < 0$ for $x \in (0, 1)$, $\ln 1 = 0$, and $\ln x > 0$ for $x \in (1, \infty)$. We also see that $\ln x$ is an INCreasing function of x.

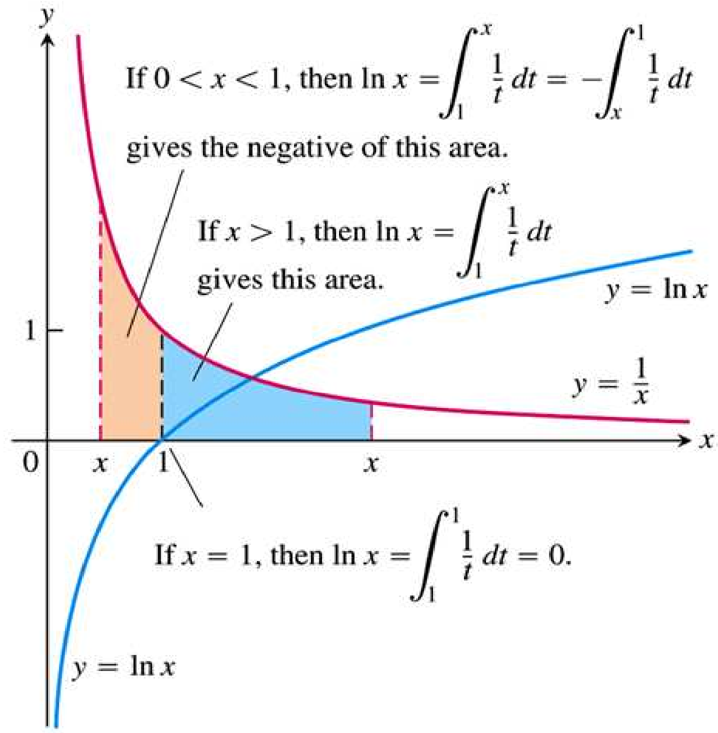

Figure 7.1 page 418

Definition. The number e is that number in the domain of the natural logarithm satisfying $\ln(e) = 1$. Numerically, $e \approx 2.718281828459045$.

Note. The number e is an example of a *transcendental number* (as opposed to an *algebraic number*). The number π is also transcendental. The text mentions this on page 422. The six trigonometric functions, the logarithm functions, and exponential functions (to be defined soon) are transcendental (thus the title of this chapter).

Theorem. For $x > 0$ we have

$$\frac{d}{dx}[\ln x] = \frac{1}{x}.$$

If $u = u(x)$ is a differentiable function of x, then for all x such that $u(x) > 0$ we have

$$\frac{d}{dx}[\ln u] = \frac{d}{dx}[\ln u(x)] = \frac{1}{u}\left[\frac{du}{dx}\right] = \frac{1}{u(x)}[u'(x)].$$

Proof. We have by the Fundamental Theorem of Calculus Part 1:

$$\frac{d}{dx}[\ln x] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x}.$$

By the Chain Rule

$$\frac{d}{dx}[\ln u(x)] = \frac{d}{du}[\ln u]\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

Q.E.D.

Note. We can use the above result to show that for $x \neq 0$, we have $\frac{d}{dx}[\ln |x|] = \frac{1}{x}$.

Examples. Page 426 numbers 14 and 26.

Theorem. Properties of Logarithms. For any numbers $b > 0$ and $x > 0$ we have

1. $\ln bx = \ln b + \ln x$
2. $\ln \dfrac{b}{x} = \ln b - \ln x$
3. $\ln \dfrac{1}{x} = -\ln x$
4. $\ln x^r = r \ln x$.

Proof. First for **1**. Notice that

$$\frac{d}{dx}[\ln bx] = \frac{1}{bx}\frac{d}{dx}[bx] = \frac{1}{bx}b = \frac{1}{x}.$$

This is the same as the derivative of $\ln x$. Therefore by Corollary 1 to the Mean Value Theorem, $\ln bx$ and $\ln x$ differ by a constant, say $\ln bx = \ln x + k_1$ for some constant k_1. By setting $x = 1$ we need $\ln b = \ln 1 + k_1 = 0 + k_1 = k_1$. Therefore $k_1 = \ln b$ and we have the identity $\ln bx = \ln b + \ln x$.

Now for **2**. We know by **1**:

$$\ln \frac{1}{x} + \ln x = \ln\left(\frac{1}{x}x\right) = \ln 1 = 0.$$

Therefore $\ln \dfrac{1}{x} = -\ln x$. Again by **1** we have

$$\ln \frac{b}{x} = \ln\left(b\frac{1}{x}\right) = \ln b + \ln \frac{1}{x} = \ln b - \ln x.$$

Finally for **4**. We have by the Chain Rule:

$$\frac{d}{dx}\left[\ln x^r\right] = \frac{1}{x^r}\frac{d}{dx}\left[x^r\right] = \frac{1}{x^r} r x^{r-1} = r\frac{1}{x} = r\frac{d}{dx}\left[\ln x\right] = \frac{d}{dx}\left[r \ln x\right].$$

As in the proof of **1**, since $\ln x^r$ and $r \ln x$ have the same derivative, we have $\ln x^r = r \ln x + k_2$ for some k_2. With $x = 1$ we see that $k_2 = 0$ and we have $\ln x^r = r \ln x$. *Q.E.D.*

Theorem. If u is a differentiable function that is never zero then

$$\int \frac{1}{u}\, du = \ln |u| + C = \{\ln |u(x)| + k \mid k \in \mathbb{R}\}.$$

Proof. We know the result holds for $u(x) > 0$. We must only establish it for $u(x) < 0$. Notice that when $u(x) < 0$, $-u(x) > 0$, and $|u(x)| = -u(x)$

$$\int \frac{1}{u}\, du = \int \frac{1}{-u} d(-u) = \ln(-u) + C = \ln |u| + C.$$

Q.E.D.

Note. We can also express the previous theorem as

$$\int \frac{1}{u(x)} u'(x)\, dx = \ln |u(x)| + C.$$

where $u(x)$ is nonzero.

Examples. Page 425 numbers 4 and 6.

Theorem. For $u = u(x)$ a differentiable function,

$$\int \tan u\, du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u\, du = \ln|\sin u| + C = -\ln|\csc u| + C.$$

Proof. Both follow from u-substitution:

$$\begin{aligned}
\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx \\
&\quad \text{Let } u = \cos x \\
&\quad \text{then } du = -\sin x\, dx \\
&= \int \frac{-du}{u} = -\int \frac{du}{u} \\
&= -\ln|u| + C = -\ln|\cos x| + C \\
&= \ln \frac{1}{|\cos x|} + C \\
&= \ln|\sec x| + C.
\end{aligned}$$

Next

$$\begin{aligned}
\int \cot x \, dx &= \int \frac{\cos x}{\sin x} \, dx \\
&\quad \text{Let } u = \sin x \\
&\quad \text{then } du = \cos x \, dx \\
&= \int \frac{du}{u} = \int \ln |u| + C \\
&= \ln |\sin x| + C = -\ln \frac{1}{|\sin x|} \\
&= -\ln |\csc x| + C.
\end{aligned}$$

The theorem follows by another application of u-substitution. *Q.E.D.*

Theorem. For $u = u(x)$ a differentiable function,

$$\int \sec u \, du = \ln |\sec u + \tan u| + C$$

$$\int \csc u \, du = -\ln |\csc u + \cot u| + C.$$

Proof. Both follow from u-substitution and a "trick." Here's the first one:

$$
\begin{aligned}
\int \sec x\,dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx \\
&\quad \text{Let } u = \sec x + \tan x \\
&\quad \text{then } du = (\sec x \tan x + \sec^2 x)\,dx \\
&= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C.
\end{aligned}
$$

The result follows by another application of u-substitution. The second result follows similarly (see page 421). *Q.E.D.*

Note. It is clear that the domain of the natural logarithm function is $(0, \infty)$. The range of the natural logarithm function is $(-\infty, \infty)$, as argued on page 420. We know that the natural logarithm function is increasing, so it is one-to-one and has an inverse.

Definition. Define the *natural exponential function* as $e^x = \ln^{-1} x = \exp x$.

Note. The domain of e^x is $(-\infty, \infty)$ (the same as the range of $\ln x$) and the range of e^x is $(0, \infty)$ (the same as the domain of $\ln x$).

Note. Since we have defined e^x as the inverse of $\ln x$, we immediately have:

$$\begin{aligned} e^{\ln x} &= x \text{ for } x \in (0, \infty) \\ \ln(e^x) &= x \text{ for all } x. \end{aligned}$$

Theorem. We have

$$\frac{d}{dx}\left[e^x\right] = e^x.$$

Proof. Let $y = e^x = \ln^{-1} x$. Then

$$\ln y = \ln e^x = x.$$

Then

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}[x]$$

$$\frac{1}{y}\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y = e^x.$$

This proof differs from that on page 422, which uses a theorem on the derivative of inverse functions (from section 3.8). *Q.E.D.*

Note. By combining the previous theorem with the Chain Rule we have

$$\frac{d}{dx}[e^u] = e^u \left[\frac{du}{dx}\right].$$

Example. Differentiate $f(x) = e^{\cos x} \sec x$.

Theorem. We have

$$\int e^u \, du = e^u + C.$$

Proof. This is just a statement of the previous theorem in integral form.

Q.E.D.

Example. Page 426 number 20.

Theorem 1. For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$.
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Note. The proof of Theorem 1 is based on the definition of $y = e^x$ in terms of $x = \ln y$ and properties of the natural logarithm function.

Definition. For any numbers $a > 0$ and x, the exponential function with base a is given by $a^x = e^{x \ln a}$.

Note. We can use the above definition to show that Theorem 1 holds for exponentials base a as well.

Theorem. If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}[a^u] = a^u \ln a \, \frac{du}{dx}.$$

In terms of integrals,

$$\int a^u \, du = \frac{a^u}{\ln a} + C.$$

Definition. For any positive number $a \neq 1$, the *logarithm of x with base a*, denoted by $\log_a x$, is the inverse function of a^x.

Note. Since we have defined a^x as the inverse of $\log_a x$, we immediately have:

$$\begin{aligned} a^{\log_a x} &= x \text{ for } x \in (0, \infty) \\ \log_a(a^x) &= x \text{ for all } x. \end{aligned}$$

Theorem. To convert logarithms from one base to another, we have the formula (which we may take as a definition):

$$\log_a x = \frac{\log_b x}{\log_b a}$$

or using natural logarithms:

$$\log_a x = \frac{\ln x}{\ln a}.$$

Proof. We set $y = \log_a x$ and so $a^y = x$. Taking natural logarithms, we have $\ln a^y = \ln x$, or $y \ln a = \ln x$ and $y = \dfrac{\ln x}{\ln a}$. Therefore, $y = \log_a x = \dfrac{\ln x}{\ln a}$. *Q.E.D.*

Note. We can use the previous conversion result and the properties of $\ln x$ to show that $\log_a x$ satisfies all the usual properties of logarithms.

Theorem. Differentiating a logarithm base a gives:

$$\frac{d}{dx}[\log_a u] = \frac{1}{\ln a}\frac{1}{u}\frac{du}{dx}.$$

Proof. This follows easily:

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{\ln a}\frac{d}{dx}[\ln x] = \frac{1}{\ln a}\frac{1}{x}.$$

Combining this result with the Chain Rule gives the theorem. *Q.E.D.*

Examples. Page 426 numbers 46 and 54.