Chapter 7. Integrals and Transcendental Functions

7.2. Exponential Change and Separable Differential Equations

Note. Suppose we are interested in a quantity y (population, radioactive element, money) that increases or decreases at a rate proportional to the amount present. If we also know the amount present at time $t = 0$, say y_0, we can find y as a function of t by solving the following initial value problem.

$$\text{Differential equation: } \frac{dy}{dt} = ky$$

$$\text{Initial condition: } \quad y = y_0 \text{ when } t = 0.$$

The constant function $y = 0$ is a solution of the differential equation but we usually aren't interested in that solution. To find nonzero solutions, we separate the variables and integrate.

$$
\begin{aligned}
\frac{dy}{y} &= k\,dt \\
\ln|y| &\in kt + C \\
e^{\ln|y|} &= e^{kt+c} \text{ where } c \in C \\
|y| &= e^c e^{kt} \text{ where } c \in C \\
y &= \pm e^c e^{kt} \\
y &= Ae^{kt} \text{ where } A = \pm e^c.
\end{aligned}
$$

By allowing A to take on the value 0 in addition to all the possible values we can include the solution $y = 0$. Therefore the general solution to the given differential equation is $y = Ae^{kt}$ where A is an arbitrary constant.

Note. If y changes at a rate proportional to the amount present ($dy/dx = ky$) and $y = y_0$ when $t = 0$, then

$$y = y_0 e^{kt},$$

where $k > 0$ represents growth and $k < 0$ represents decay. This is call the *Law of Exponential Change* and k is the *rate constant* of the equation.

Example. Page 434 numbers 30 and 32.

Definition. A *first-order differential equation* is a relation

$$\frac{d}{dx}[y(x)] = f(x, y(x))$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. A *solution* of this equation is a differentiable function $y = y(x)$ defined on an interval of x-values such that

$$\frac{d}{dx}[y(x)] = f(x, y(x))$$

on that interval. The initial condition that $y(x_0) = y_0$ amounts to requiring the solution curve $y = y(x)$ to pass though the point (x_0, y_0).

Example. Page 434 number 6.

Definition. The equation $y' = f(x, y)$ is *separable* if f can be expressed as a product of a function of x and a functions of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y) = \frac{g(x)}{h(y)}$$

where $H(y) = \dfrac{1}{h(y)}$.

Note. If $h(y) \neq 0$, we can *separate the variables* by dividing both sides by h, obtaining

$$\begin{aligned} \frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\ \int \frac{1}{h(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\ \int \frac{1}{h(y)}\,dy &= \int g(x)\,dx \end{aligned}$$

(Notice that the last two lines claim that two *sets* are equal.) With x and y now separated, we simply integrate each side to get the solutions. We seek by expressing y either explicitly or implicitly as a function of x, up to an arbitrary constant.

Example. Page 434 number 16.

Note. When an atom emits some of its mass as radiation, the remainder of the atom reforms to make an atom of some new element. This process of radiation and change is *radioactive decay*, and an element whose atoms go spontaneously through this process is *radioactive*. For example, radioactive carbon-14 decays into nitrogen.

Experiments have shown that at any given time, the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit of time) is approximately proportional to the number of radioactive nuclei present. Thus, the decay of a radioactive element is described by the equation $dy/dt = -ky$, $k > 0$. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time will be

$$y = y_0 e^{-kt}, k > 0.$$

Definition. The *half-life* of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay.

Note. We can calculate half-life by asking $t =?$ when $y = y_0/2$. This gives $y_0/2 = y_0 e^{-kt}$ or $1/2 = e^{-kt}$. By taking logarithms of both sides of this last equation, we get $\ln(1/2) = \ln e^{-kt} = -kt$. From this it follows that $t = -(\ln 1/2)/k = (\ln 2)/k$.

Example. The following radioactive isotopes are commonly used for determining ages of rocks:

Isotope	**Half-Life**	**Daughter Product**
K-40	1.3 billion years	Ar-40
U-238	4.5 billion years	Pb-206
U-235	713 million years	Pb-207
Th-232	14.1 billion years	Pb-208
Rb-87	49 billion years	Sr-87

When a rock is formed, it contains U-238 and no Pb-206. After some time, the rock contains only 49% of the original amount of U-238 (the rest having decayed into Pb-206). How old is the rock?

Note. If H is the temperature of an object in an environment of temperature H_S, then according to *Newton's Law of Cooling*, the objects temperature changes at a rate proportional to the difference of H and H_S. That is,

$$\frac{dH}{dt} = -k(H - H_S).$$

If H_0 is the initial temperature of the object, then we get

$$H - H_S = (H_0 - H_S)e^{-kt}.$$

Example. Page 435 number 42.